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245:00: |a Analytic geometry, |c by L. Wayland Dowling and F. E. Turneaure.
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300/1: : |a 2 p. L., iii-xii, 266 p. |b diagrs. |c 21 cm.
490/1:0 : |a American mathematical series. E. J. Townsend, general ed. |v 8
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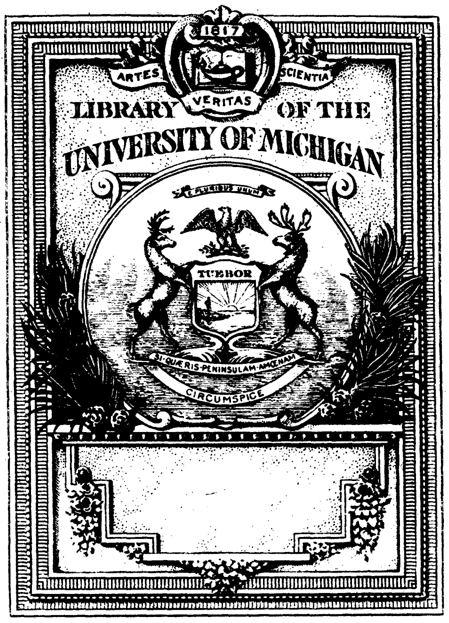

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American Mathematical Series

E. J. TOWNSEND

GENERAL EDITOR

ANALYTIC GEOMETRY

BY

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AND

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"Exhibit Collection of Mathematical Text Books"

NEW YORK
HENRY HOLT AND COMPANY

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3

Norwood Press
J. S. Cushing Co. — Berwick & Smith Co.
Norwood, Mass., U.S.A.

PREFACE

In accordance with the general plan of this series of textbooks, the authors of the present volume have had constantly in mind the needs of the student who takes his mathematics primarily with a view to its applications as well as the needs of the student who pursues mathematics as an element of his education.

The processes of analytical geometry find their application, for the most part, in the scientific laboratory where it is often necessary to study the properties of a function from certain observed values. The fundamental concept is, therefore, that of functional correspondence and the methods of representing such correspondence geometrically. For this reason rather more than usual attention has been given to these subjects (Chapter III; also Chapter IX, Arts. 135 to 140).

An intelligent appreciation of functional correspondence requires an intimate knowledge of the relation between an equation and the graphical representation of the functional correspondence determined by the equation. Such a knowledge is most easily obtained by a study of linear equations and equations of the second degree together with their corresponding loci. This knowledge is not only of importance to the student of applied mathematics, but it has a special disciplinary value for the general student.

The standard forms of the equations of a number of important loci are developed early (Chapter IV), and the properties of these loci are discussed in detail later (Chapters VI and VII) by means of the equations already at hand. By this arrangement, it is hoped that some unnecessary repetition has been avoided.

The equations of tangents to the conic sections have been derived by means of the discriminant of the quadratic equation whose roots are the x-coördinates of the points of intersection with a variable secant, rather than by means of the derivative. This course has been adopted, first, because the geometric interpretation of the discriminant is important in itself; and, second, because the use of the derivative ought, logically, to be preceded by a chapter devoted to its definition and the methods for finding it, at least for algebraic functions. Moreover, the use of the derivative for finding the equations of tangents is only one of its many applications. No student should feel that his mathematical education is complete without a knowledge of the calculus, where he will become familiar with the derivative and can appreciate its usefulness in many directions.

The present volume is designed for a four-hour, or a five-hour, course for one semester, but may be shortened to a three-hour course by omitting certain parts of the text. For example, Art. 105 may be omitted without marring the continuity of the course. Again, Arts. 110, 111, and 112 contain all that is essential in dealing with the general equation of the second degree in two variables, and the remainder of Chapter VIII can therefore be omitted from the longer course. Parts of Chapter IX can also be omitted according to the needs of the student. The chapters on solid analytic geometry have been added for the benefit of those students who have time only for an outline of the subject matter. No apology is therefore offered for the meager treatment.

The authors desire to express their appreciation to their colleagues of the University of Wisconsin and of the University of Illinois for the assistance and the many helpful suggestions given them during the preparation of the book. They are under especial obligations to Professor W. H. Bussey, of the University of Minnesota; Professor S. C. Davisson, of the University of Indiana; Professor J. L. Markley, of the University of Michigan; and Professor E. J. Townsend, of the University of Illinois, for their care and assistance in seeing the book through the press.

L. W. DOWLING,
F. E. TURNEAURE.

UNIVERSITY OF WISCONSIN,
July, 1914.

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GREEK ALPHABET

Letters		Names	Letters		Names
Capitals	Lower Case		Capitals	Lower Case	
Α	α	Alpha	Ν	ν	Nu
Β	β	Beta	Ξ	ξ	Xi
Γ	γ	Gamma	Ο	ο	Omicron
Δ	δ	Delta	Π	π	Pi
Ε	ϵ	Epsilon	Ρ	ρ	Rho
Ζ	ζ	Zeta	Σ	σ	Sigma
Η	η	Eta	Τ	τ	Tau
Θ	θ	Theta	Υ	υ	Upsilon
Ι	ι	Iota	Φ	ϕ	Phi
Κ	κ	Kappa	Χ	χ	Chi
Λ	λ	Lambda	Ψ	ψ	Psi
Μ	μ	Mu	Ω	ω	Omega

ANALYTIC GEOMETRY

INTRODUCTION

A. The quadratic equation. For convenience in reference the following formulas and tables are introduced.

Any quadratic equation may be written in the form

$$ax^2 + bx + c = 0.$$

The two roots of this equation are

$$\boldsymbol{x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}}, \text{ and } \boldsymbol{x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}}.$$

By addition, $$x_1 + x_2 = -\frac{b}{a}.$$

By multiplication, $$x_1x_2 = \frac{c}{a}.$$

The sum and the product of the roots can therefore be found directly from the equation without solving.

The character of the roots depends on the quantity under the radical, $b^2 - 4ac$.

If $\boldsymbol{b^2 - 4ac > 0}$, the roots are real and unequal,
if $\boldsymbol{b^2 - 4ac = 0}$, the roots are real and equal,
if $\boldsymbol{b^2 - 4ac < 0}$, the roots are imaginary.

The expression $b^2 - 4ac$ is called the **discriminant** of the equation, and when placed equal to zero expresses the condition which must hold between the coefficients, if the two roots of the equation are equal.

B. Trigonometric formulas. If A, B, and C are the angles of a triangle and a, b, c are respectively the lengths of the sides opposite, then :

(1) *Law of sines:*

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}.$$

(2) *Law of cosines:*

$$a^2 = b^2 + c^2 - 2\,bc \cos A,$$
$$b^2 = a^2 + c^2 - 2\,ac \cos B,$$
$$c^2 = a^2 + b^2 - 2\,ab \cos C.$$

(3) *Law of tangents:*

$$\frac{\tan \frac{1}{2}(A + B)}{\tan \frac{1}{2}(A - B)} = \frac{a + b}{a - b}.$$

Addition formulas. If A and B are any angles, then

$$\sin(A \pm B) = \sin A \cos B \pm \sin B \cos A,$$
$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B,$$
$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}.$$

C. TABLES

Common Logarithms

N	0	D	1	D	2	D	3	D	4	D	5	D	6	D	7	D	8	D	9	D
10	0000	43	0043	43	0086	42	0128	42	0170	42	0212	41	0253	41	0294	40	0334	40	0374	40
11	0414	39	0453	39	0492	39	0531	38	0569	38	0607	38	0645	37	0682	37	0719	36	0755	37
12	0792	36	0828	36	0864	35	0899	35	0934	35	0969	35	1004	34	1038	34	1072	34	1106	33
13	1139	34	1173	33	1206	33	1239	32	1271	32	1303	32	1335	32	1367	32	1399	31	1430	31
14	1461	31	1492	31	1523	30	1553	31	1584	30	1614	30	1644	29	1673	30	1703	29	1732	29
15	1761	29	1790	28	1818	29	1847	28	1875	28	1903	28	1931	28	1959	28	1987	27	2014	27
16	2041	27	2068	27	2095	27	2122	26	2148	27	2175	26	2201	26	2227	26	2253	26	2279	25
17	2304	26	2330	25	2355	25	2380	25	2405	25	2430	25	2455	25	2480	24	2504	25	2529	24
18	2553	24	2577	24	2601	24	2625	23	2648	24	2672	23	2695	23	2718	24	2742	23	2765	23
19	2788	22	2810	23	2833	23	2856	22	2878	22	2900	23	2923	22	2945	22	2967	22	2989	21
20	3010	22	3032	22	3054	21	3075	21	3096	22	3118	21	3139	21	3160	21	3181	20	3201	21
21	3222	21	3243	20	3263	21	3284	20	3304	20	3324	21	3345	20	3365	20	3385	19	3404	20
22	3424	20	3444	20	3464	19	3483	19	3502	20	3522	19	3541	19	3560	19	3579	19	3598	19
23	3617	19	3636	19	3655	19	3674	18	3692	19	3711	18	3729	18	3747	19	3766	18	3784	18
24	3802	18	3820	18	3838	18	3856	18	3874	18	3892	17	3909	18	3927	18	3945	17	3962	17
25	3979	18	3997	17	4014	17	4031	17	4048	17	4065	17	4082	17	4099	17	4116	17	4133	17
26	4150	16	4166	17	4183	17	4200	16	4216	16	4232	17	4249	16	4265	16	4281	17	4198	16
27	4314	16	4330	16	4346	16	4362	16	4378	15	4393	16	4409	16	4425	15	4440	16	4456	16
28	4472	15	4487	15	4502	16	4518	15	4533	15	4548	16	4564	15	4579	15	4594	15	4609	15
29	4624	15	4639	15	4654	15	4669	14	4683	15	4698	15	4713	15	4728	14	4742	15	4757	14
30	4771	15	4786	14	4800	14	4814	15	4829	14	4843	14	4857	14	4871	15	4886	14	4900	14
31	4914	14	4928	14	4942	13	4955	14	4969	14	4983	14	4997	14	5011	13	5024	14	5038	13
32	5051	14	5065	14	5079	13	5092	13	5105	14	5119	13	5132	13	5145	14	5159	13	5172	13
33	5185	13	5198	13	5211	13	5224	13	5237	13	5250	13	5263	13	5276	13	5289	13	5302	13
34	5315	13	5328	12	5340	13	5353	13	5366	12	5378	13	5391	12	5403	13	5416	12	5428	13
35	5441	12	5453	12	5465	13	5478	12	5490	12	5502	12	5514	13	5527	12	5539	12	5551	12
36	5563	12	5575	12	5587	12	5599	12	5611	12	5623	12	5635	12	5647	11	5658	12	5670	12
37	5682	12	5694	11	5705	12	5717	12	5729	11	5740	12	5752	11	5763	12	5775	11	5786	12
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45	6532	10	6542	9	6551	10	6561	10	6571	9	6580	10	6590	9	6599	10	6609	9	6618	10
46	6628	9	6637	9	6646	10	6656	9	6665	10	6675	9	6684	9	6693	9	6702	10	6712	9
47	6721	9	6730	9	6739	10	6749	9	6758	9	6767	9	6776	9	6785	9	6794	9	6803	9
48	6812	9	6821	9	6830	9	6839	9	6848	9	6857	9	6866	9	6875	9	6884	9	6893	9
49	6902	9	6911	9	6920	8	6928	9	6937	9	6946	9	6955	9	6964	8	6972	9	6981	9
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51	7076	8	7084	9	7093	8	7101	9	7110	8	7118	8	7126	9	7135	8	7143	9	7152	8
52	7160	8	7168	9	7177	8	7185	8	7193	9	7202	8	7210	8	7218	8	7226	9	7235	8
53	7243	8	7251	8	7259	8	7267	8	7275	9	7284	8	7292	8	7300	8	7308	8	7316	8
54	7324	8	7332	8	7340	8	7348	8	7356	8	7364	8	7372	8	7380	8	7388	8	7396	8

Common Logarithms — Continued

N	0	D	1	D	2	D	3	D	4	D	5	D	6	D	7	D	8	D	9	D
55	7404	8	7412	7	7419	8	7427	8	7435	8	7443	8	7451	8	7459	7	7466	8	7474	8
56	7482	8	7490	7	7497	8	7505	8	7513	7	7520	8	7528	8	7536	7	7543	8	7551	8
57	7559	7	7566	8	7574	8	7582	7	7589	8	7597	7	7604	8	7612	7	7619	8	7627	7
58	7634	8	7642	7	7649	8	7657	7	7664	8	7672	7	7679	7	7686	8	7694	7	7701	8
59	7709	7	7716	7	7723	8	7731	7	7738	7	7745	7	7752	8	7760	7	7767	7	7774	8
60	7782	7	7789	7	7796	7	7803	7	7810	8	7818	7	7825	7	7832	7	7839	7	7846	7
61	7853	7	7860	8	7868	7	7875	7	7882	7	7889	7	7896	7	7903	7	7910	7	7917	7
62	7924	7	7931	7	7938	7	7945	7	7952	7	7959	7	7966	7	7973	7	7980	7	7987	6
63	7993	7	8000	7	8007	7	8014	7	8021	7	8028	7	8035	6	8041	7	8048	7	8055	7
64	8062	7	8069	6	8075	7	8082	7	8089	7	8096	6	8102	7	8109	7	8116	6	8122	7
65	8129	7	8136	6	8142	7	8149	7	8156	6	8162	7	8169	7	8176	6	8182	7	8189	6
66	8195	7	8202	7	8209	6	8215	7	8222	6	8228	7	8235	6	8241	7	8248	6	8254	7
67	8261	6	8267	7	8274	6	8280	7	8287	6	8293	6	8299	7	8306	6	8312	7	8319	6
68	8325	6	8331	7	8338	6	8344	7	8351	6	8357	6	8363	7	8370	6	8376	6	8382	6
69	8388	7	8395	6	8401	6	8407	7	8414	6	8420	6	8426	6	8432	7	8439	6	8445	6
70	8451	6	8457	6	8463	7	8470	6	8476	6	8482	6	8488	6	8494	6	8500	6	8506	7
71	8513	6	8519	6	8525	6	8531	6	8537	6	8543	6	8549	6	8555	6	8561	6	8567	6
72	8573	6	8579	6	8585	6	8591	6	8597	6	8603	6	8609	6	8615	6	8621	6	8627	6
73	8633	6	8639	6	8645	6	8651	6	8657	6	8663	6	8669	6	8675	6	8681	5	8686	6
74	8692	6	8698	6	8704	6	8710	6	8716	6	8722	5	8727	6	8733	6	8739	6	8745	6
75	8751	5	8756	6	8762	6	8768	6	8774	5	8779	6	8785	6	8791	6	8797	5	8802	6
76	8808	6	8814	6	8820	5	8825	6	8831	6	8837	5	8842	6	8848	6	8854	5	8859	6
77	8865	6	8871	5	8876	6	8882	5	8887	6	8893	6	8899	5	8904	6	8910	5	8915	6
78	8921	6	8927	5	8932	6	8938	5	8943	6	8949	5	8954	6	8960	5	8965	6	8971	5
79	8976	6	8982	5	8987	6	8893	5	8998	6	9004	5	9009	6	9015	5	9020	5	9025	6
80	9031	5	9036	6	9042	5	9047	6	9053	5	9058	5	9063	6	9069	5	9074	5	9079	6
81	9085	5	9090	6	9096	5	9101	5	9106	6	9112	5	9117	5	9122	6	9128	5	9133	5
82	9138	5	9143	6	9149	5	9154	5	9159	6	9165	5	9170	5	9175	5	9180	6	9186	5
83	9191	5	9196	5	9201	5	9206	6	9212	5	9217	5	9222	5	9227	5	9232	6	9238	5
84	9243	5	9248	5	9253	5	9258	5	9263	6	9269	5	9274	5	9279	5	9284	5	9289	5
85	9294	5	9299	5	9304	5	9309	6	9315	5	9320	5	9325	5	9330	5	9335	5	9340	5
86	9345	5	9350	5	9355	5	9360	5	9365	5	9370	5	9375	5	9380	5	9385	5	9390	5
87	9395	5	9400	5	9405	5	9410	5	9415	5	9420	5	9425	5	9430	5	9435	5	9440	5
88	9445	5	9450	5	9455	5	9460	5	9465	4	9469	5	9474	5	9479	5	9484	5	9489	5
89	9494	5	9499	5	9504	5	9509	4	9513	5	9518	5	9523	5	9528	5	9533	5	9538	4
90	9542	5	9547	5	9552	5	9557	5	9562	4	9566	5	9571	5	9576	5	9581	5	9586	4
91	9590	5	9595	5	9600	5	9605	4	9609	5	9614	5	9619	5	9624	4	9628	5	9633	5
92	9638	5	9643	4	9647	5	9652	5	9657	4	9661	5	9666	5	9671	4	9675	5	9680	5
93	9685	4	9689	5	9694	5	9699	4	9703	5	9708	5	9713	4	9717	5	9722	5	9727	4
94	9731	5	9736	5	9741	4	9745	5	9750	4	9754	5	9759	4	9763	5	9768	5	9773	4
95	9777	5	9782	4	9786	5	9791	4	9795	5	9800	5	9805	4	9809	5	9814	4	9818	5
96	9823	4	9827	5	9832	4	9836	5	9841	4	9845	5	9850	4	9854	5	9859	4	9863	5
97	9868	4	9872	5	9877	4	9881	5	9886	4	9890	4	9894	5	9899	4	9903	5	9908	4
98	9912	5	9917	4	9921	5	9926	4	9930	4	9934	5	9939	4	9943	5	9948	4	9952	4
99	9956	5	9961	4	9965	4	9969	5	9974	4	9978	5	9983	4	9987	4	9991	5	9996	4

Trigonometric Functions

[Characteristics of Logarithms omitted — determine by the usual rule from the value]

Radians	Degrees	Sine		Tangent		Cotangent		Cosine			
		Value	log$_{10}$	Value	log$_{10}$	Value	log$_{10}$	Value	log$_{10}$		
.0000	0°	.0000	$-\infty$	.0000	$-\infty$	∞	∞	1.0000	0000	90°	1.5708
.0175	1°	.0175	2419	.0175	2419	57.290	7581	.9998	9999	89°	1.5533
.0349	2°	.0349	5428	.0349	5431	28.636	4569	.9994	9997	88°	1.5359
.0524	3°	.0523	7188	.0524	7196	19.081	2806	.9986	9994	87°	1.5184
.0698	4°	.0698	8436	.0699	8448	14.301	1554	.9976	9989	86°	1.5010
.0873	5°	.0872	9403	.0875	9420	11.430	0580	.9962	9983	85°	1.4835
.1047	6°	.1045	0192	.1051	0216	9.5144	9784	.9945	9976	84°	1.4661
.1222	7°	.1219	0859	.1228	0891	8.1443	9109	.9925	9968	83°	1.4486
.1396	8°	.1392	1436	.1405	1478	7.1154	8522	.9903	9958	82°	1.4312
.1571	9°	.1564	1943	.1584	1997	6.3138	8003	.9877	9946	81°	1.4137
.1745	10°	.1736	2397	.1763	2463	5.6713	7537	.9848	9934	80°	1.3963
.1920	11°	.1908	2806	.1944	2887	5.1446	7113	.9816	9919	79°	1.3788
.2094	12°	.2079	3179	.2126	3275	4.7046	6725	.9781	9904	78°	1.3614
.2269	13°	.2250	3521	.2309	3634	4.3315	6366	.9744	9887	77°	1.3439
.2443	14°	.2419	3837	.2493	3968	4.0108	6032	.9703	9869	76°	1.3265
.2618	15°	.2588	4130	.2679	4281	3.7321	5719	.9659	9849	75°	1.3090
.2793	16°	.2756	4403	.2867	4575	3.4874	5425	.9613	9828	74°	1.2915
.2967	17°	.2924	4659	.3057	4853	3.2709	5147	.9563	9806	73°	1.2741
.3142	18°	.3090	4900	.3249	5118	3.0777	4882	.9511	9782	72°	1.2566
.3316	19°	.3256	5126	.3443	5370	2.9042	4630	.9455	9757	71°	1.2392
.3491	20°	.3420	5341	.3640	5611	2.7475	4389	.9397	9730	70°	1.2217
.3665	21°	.3584	5543	.3839	2842	2.6051	4158	.9336	9702	69°	1.2043
.3840	22°	.3746	5736	.4040	6064	2.4751	3936	.9272	9672	68°	1.1868
.4014	23°	.3907	5919	.4245	6279	2.3559	3721	.9205	9640	67°	1.1694
.4189	24°	.4067	6093	.4452	6486	2.2460	3514	.9135	9607	66°	1.1519
.4363	25°	.4226	6259	.4663	6687	2.1445	3313	.9063	9573	65°	1.1345
.4538	26°	.4384	6418	.4877	6882	2.0503	3118	.8988	9537	64°	1.1170
.4712	27°	.4540	6570	.5095	7072	1.9626	2928	.8910	9499	63°	1.0996
.4887	28°	.4695	6716	.5317	7257	1.8807	2743	.8829	9459	62°	1.0821
.5061	29°	.4848	6856	.5543	7438	1.8040	2562	.8746	9418	61°	1.0647
.5236	30°	.5000	6990	.5774	7614	1.7321	2386	.8660	9375	60°	1.0472
.5411	31°	.5150	7118	.6009	7788	1.6643	2212	.8572	9331	59°	1.0297
.5585	32°	.5299	7242	.6249	7958	1.6003	2042	.8480	9284	58°	1.0123
.5760	33°	.5446	7361	.6494	8125	1.5399	1875	.8387	9236	57°	.9948
.5934	34°	.5592	7476	.6745	8290	1.4826	1710	.8290	9186	56°	.9774
.6109	35°	.5736	7586	.7002	8452	1.4281	1548	.8192	9134	55°	.9599
.6283	36°	.5878	7692	.7265	8613	1.3764	1387	.8090	9080	54°	.9425
.6458	37°	.6018	7795	.7536	8771	1.3270	1229	.7986	9023	53°	.9250
.6632	38°	.6157	7893	.7813	8928	1.2799	1072	.7880	8965	52°	.9076
.6807	39°	.6293	7989	.8098	9084	1.2349	0916	.7771	8905	51°	.8901
.6981	40°	.6428	8081	.8391	9238	1.1918	0762	.7660	8843	50°	.8727
.7156	41°	.6561	8169	.8693	9392	1.1504	0608	.7547	8778	49°	.8552
.7330	42°	.6691	8255	.9004	9544	1.1106	0456	.7431	8711	48°	.8378
.7505	43°	.6820	8338	.9325	9697	1.0724	0303	.7314	8641	47°	.8203
.7679	44°	.6947	8418	.9657	9848	1.0355	0152	.7193	8569	46°	.8029
.7854	45°	.7071	8495	1.0000	0000	1.0000	0000	.7071	8495	45°	.7854
		Value	log$_{10}$	Value	log$_{10}$	Value	log$_{10}$	Value	log$_{10}$	Degrees	Radians
		Cosine		Cotangent		Tangent		Sine			

Exponential Functions

		e^x		e^{-x}				e^x		e^{-x}	
x	$\log_e x$	Value	$\log_{10}$	Value	$\log_{10}$	x	$\log_e x$	Value	$\log_{10}$	Value	$\log_{10}$
0.0	$-\infty$	1.000	0.000	1.000	0.000	2.0	0.693	7.389	0.869	0.135	9.131
0.1	−2.303	1.105	0.043	0.905	9.957	2.1	0.742	8.166	0.912	0.122	9.088
0.2	−1.610	1.221	0.087	0.819	9.913	2.2	0.788	9.025	0.955	0.111	9.045
0.3	−1.204	1.350	0.130	0.741	9.870	2.3	0.833	9.974	0.999	0.100	9.001
0.4	−0.916	1.492	0.174	0.670	9.826	2.4	0.875	11.02	1.023	0.091	8.958
0.5	−0.693	1.649	0.217	0.607	9.783	2.5	0.916	12.18	1.086	0.082	8.914
0.6	−0.511	1.822	0.261	0.549	9.739	2.6	0.956	13.46	1.129	0.074	8.871
0.7	−0.357	2.014	0.304	0.497	9.696	2.7	0.993	14.88	1.173	0.067	8.827
0.8	−0.223	2.226	0.347	0.449	9.653	2.8	1.030	16.44	1.216	0.061	8.784
0.9	−0.105	2.460	0.391	0.407	9.609	2.9	1.065	18.17	1.259	0.055	8.741
1.0	0.000	2.718	0.434	0.368	9.566	3.0	1.099	20.09	1.303	0.050	8.697
1.1	0.095	3.004	0.478	0.333	9.522	3.5	1.253	33.12	1.520	0.030	8.480
1.2	0.182	3.320	0.521	0.301	9.479	4.0	1.386	54.60	1.737	0.018	8.263
1.3	0.262	3.669	0.565	0.273	9.435	4.5	1.504	90.02	1.954	0.011	8.046
1.4	0.336	4.055	0.608	0.247	9.392	5.0	1.609	148.4	2.171	0.007	7.829
1.5	0.405	4.482	0.651	0.223	9.349	6.0	1.792	403.4	2.606	0.002	7.394
1.6	0.470	4.953	0.695	0.202	9.305	7.0	1.946	1096.6	3.040	0.001	6.960
1.7	0.531	5.474	0.738	0.183	9.262	8.0	2.079	2981.0	3.474	0.000	6.526
1.8	0.588	6.050	0.782	0.165	9.218	9.0	2.197	8103.1	3.909	0.000	6.091
1.9	0.642	6.686	0.825	0.150	9.175	10.0	2.303	22026.	4.343	0.000	5.657

$$\log_e x = (\log_{10} x) \div M; \quad M = .4342944819 \cdot \cdot \cdot \cdot \cdot$$

PART I

PLANE ANALYTIC GEOMETRY

CHAPTER I

SYSTEMS OF COÖRDINATES

1. The linear scale. Analytic geometry is based upon a geometric representation of numbers.

Choose a straight line AB of indefinite length and upon it a fixed point O. With a convenient unit lay off the equal distances OP_1, P_1P_2, P_2P_3, $\cdots$ to the right, and OQ_1, Q_1Q_2, Q_2Q_3, $\cdots$ to the left. We will now agree that the points P_1, P_2, P_3, $\cdots$ shall represent the positive integers 1, 2, 3, $\cdots$, respectively, and the points Q_1, Q_2, Q_3, $\cdots$ shall represent the negative integers -1, -2, -3, $\cdots$, respectively.

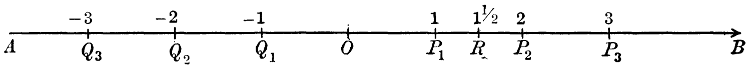


Fig. 1

The intervals along the line AB can be divided into fractional parts of the unit, thus obtaining points representing fractional numbers. For example, the point R bisecting the segment P_1P_2 represents the positive number 1.5.

The subdivision can be carried on indefinitely and we may infer, finally, that the following statement and its converse hold concerning this representation.

Each of the points on the line AB represents a number; namely, that number which expresses the distance and direction of the point from O in terms of the unit chosen.

Conversely, *if x is a positive* (*or negative*) *number, it is represented by a point x units to the right* (*or left*) *of O.*

The line AB, together with the points constructed as explained, is called a **linear scale**. It is the geometric equivalent, or graphic representation, of the system of real numbers.

The point O is called the **origin**; it represents the number zero.

The scale on a thermometer is an example of a linear scale. Here the points on the scale represent the numbers expressing degrees of temperature.

EXERCISES

1. Construct a linear scale using half an inch for the unit. On this scale, mark the points representing 3, $\frac{1}{2}$, -2, $-2\frac{3}{4}$.

2. Is the scale on an ordinary carpenter's square a linear scale? Where is the origin?

3. If the origin of the scale is moved two points to the left, how will this affect the numbers represented by the scale? If the origin is moved two points to the right, how will the numbers be affected?

4. The freezing and boiling points on a Fahrenheit thermometer are at 32° and 212° respectively, while on a centigrade thermometer they are placed at 0° and 100°. Compare the units of these two scales. Five degrees below zero on the centigrade is equivalent to what reading on the Fahrenheit? Construct the two scales in this exercise.

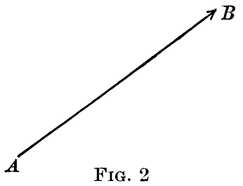


FIG. 2

2. Directed segments, directed angles. It is frequently necessary to distinguish between the two directions in which a segment may be laid off on a given straight line. This is done by calling one direction positive and the other negative. Thus, if we agree to call the direction from A to B (Fig. 2) positive, then we shall call the direction from B to A negative. Expressed in symbols,

$$AB = -BA.$$

Segments of a straight line to which a direction has been attached are called **directed segments.**

In a similar way, an angle can be directed. For example, the acute angle

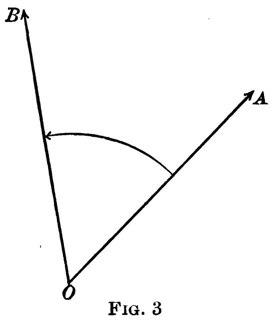


FIG. 3

shown in Fig. 3 can be described by a line rotating with the hands of a clock, or *clockwise*, from OB to OA; or *counterclockwise*, from OA to OB. It is customary to consider counterclockwise rotation as **positive**, and clockwise rotation as **negative**. Thus, the acute angle AOB is considered as a positive angle and the acute angle BOA, as a negative angle. In symbols,

$$AOB = -BOA.$$

3. Addition of directed segments, addition of directed angles. If AB, BC, and AC are three directed segments along the same line (Fig. 4), then the equation

$$AB + BC = AC$$

FIG. 4

has the following simple interpretation; namely, a point moving along the line from A to B and then from B to C is in the same final position as it would have been had it moved directly from A to C. With this interpretation, the above equation is readily seen to hold however the points A, B, and C are situated with respect to each other.

Again, we have

$$AC - AB = AC + BA = BA + AC = BC.$$

Similarly, the equation

$$AOB + BOC = AOC \text{ (Fig. 5)}$$

is interpreted as follows; rotation through the angle AOB and then through the angle BOC is equivalent to rotation through the angle AOC.

Again,

$$AOC - BOC = AOC + COB = AOB.$$

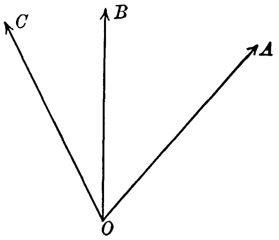


FIG. 5

EXERCISES

1. Construct a linear scale, using half an inch for the unit, and mark the points A, five units to the right of the origin, and B, three units to the left of the origin. State the geometric meaning of $OA - OB$, of $OB - OA$, of $OA + AB$. What directed segment is equivalent to each? What is the numerical value of each? Is there a directed segment in the figure equivalent to $OA + OB$?

2. What is the difference in absolute temperature between $-5°$ Fahrenheit and 20° centigrade?

3. Represent geometrically the difference in time between 10 A.M. and 3 P.M. as the difference between two directed angles.

4. In surveying, the azimuth of a line is its direction expressed in degrees, measured from the South point around towards the West, or clockwise. Thus the azimuth of a line running due North is 180°; of a line running due East is 270°; etc.

What is the azimuth of a line running N 25° E? Of a line running N 10° W?

5. What is the difference in azimuth between two lines, one running S 40° W and the other S 10° E?

6. What is the difference in azimuth between two lines, one running S 40° E and the other N 25° E?

4. Position of a point in a plane. When we have once chosen a unit of distance, one number is sufficient to locate a point on a line; namely, the number expressing its distance and direction from a fixed point, the origin (Art. 1).

It requires two numbers to locate a point in a plane and these numbers are called the **coördinates** of the point.

The coördinates of a point may be chosen in many different ways. Any particular way of choosing them gives rise to a **system of coördinates.** There are two systems of coördinates in common use; namely, cartesian coördinates, named after René Descartes, who first used this system (1637), and polar coördinates. These systems of coördinates will be explained in the succeeding articles.

5. Cartesian coördinates. Choose two linear scales OX and OY (Fig. 6) with their origins coinciding at O. Through any point P in the plane XOY draw parallels to OX and OY, meeting OY and OX in E and D, respectively. The numbers represented by D and E on their respective scales are the **cartesian coördinates** of P. By article 1, these coördinates express the distances and directions of D and E from O in terms of the unit chosen; or the distances and directions of P from OY and OX measured along parallels to OX and OY, respectively. Thus, if x and y denote the numbers represented by the points D and E, respectively, the coördinates of P are

$$x = OD = EP$$

and $$y = OE = DP.$$

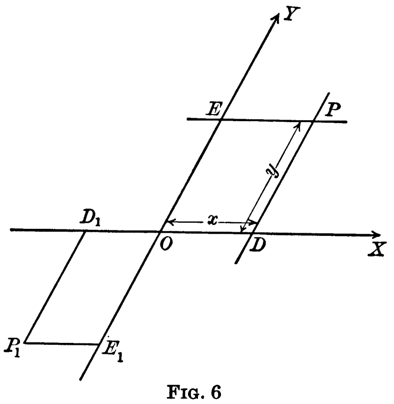


Fig. 6

In the same way, the coördinates of P_1 are

$$x_1 = OD_1 = E_1P_1 \text{ and } y_1 = OE_1 = D_1P_1.$$

Hence any point in the plane XOY has an x-coördinate and a y-coördinate represented by points on the scales OX and OY, respectively.

Conversely, any two numbers x and y serve to locate a point in the plane. For, let D and E be the points representing x and y upon their respective scales. Through D draw a line parallel to OY, and through E a line parallel to OX. These parallels meet in a single point P; the point whose coördinates are x and y.

The scales OX and OY are called the **coördinate axes**, or simply the axes. OX is called the **X-axis**, and OY the **Y-axis**.

The segment OD is often called the **abscissa** of the point P; and the segment DP the **ordinate** of P.

The units of distance used in constructing the two linear scales OX and OY are usually taken to be the same, but it is not necessary to take them so. In many cases it is more convenient to use different units.

6. Rectangular coördinates. The coördinate axes may intersect at any angle, but it is generally simpler to take them perpendicular to each other. In this case, cartesian coördinates are called **rectangular coördinates.**

In rectangular coördinates, the axes divide the plane into four quadrants named first, second, third, and fourth quadrant as indicated in Fig. 7.

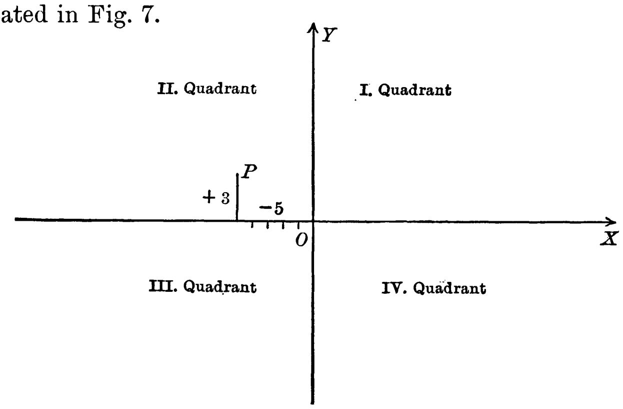


FIG. 7

The algebraic signs of the coördinates of any point depend upon the quadrant in which the point lies. Thus,

POINTS IN QUADRANT	x	y
I	+	+
II	−	+
III	−	−
IV	+	−

Conversely, the algebraic signs of the coördinates determine the quadrant in which the point lies. For example, if $x = -5$ and $y = 3$, the point lies in the second quadrant as indicated in the figure.

7. Notation. The notation $P \equiv (a, b)$, or $P(a, b)$, indicates that the coördinates of the point P are

$$x = a \text{ and } y = b.$$

The x-coördinate is always written first. For example, to indicate the position of the point P in Fig. 7, we write $P \equiv (-5, 3)$, or $P(-5, 3)$.

EXERCISES

1. Draw the axes OX, OY and locate the following points: $(\frac{1}{2}, 3)$, $(2, -\frac{3}{2})$, $(0, 5)$, $(5, 0)$.

2. Where are the points located for which $x = 0$? For which $x = 1$? For which $y = 0$? For which $y = -1$?

3. By means of a geometrical construction, locate accurately the points $(\sqrt{2}, 3)$, $(\sqrt{3}, \sqrt{2})$, $(\sqrt{5}, \sqrt{6})$. Can the point $(0, \pi)$ be located accurately?

4. The axes OX, OY are perpendicular to each other; locate the points $P_1 \equiv (1, 2)$, $P_2 \equiv (5, 5)$, and $P_3 \equiv (5, 2)$. Find the lengths of the sides of the triangle $P_1P_2P_3$.

5. Let the axes OX, OY make an angle of 60 degrees with each other; plot the points in the preceding exercise and find the lengths of the sides of the triangle $P_1P_2P_3$.

6. With rectangular coördinates, show that the points $(2, 3)$, $(2, -1)$, $(-2, -1)$, and $(-2, 3)$ form a rectangle. Find the lengths of the sides, the lengths of the diagonals, and the area of this rectangle.

7. With rectangular coördinates, show that the points $(1, 1)$, $(3, 1)$, and $(2, 2)$ form an isosceles triangle which is half a square. Find the coördinates of the fourth vertex, the lengths of the sides, the lengths of the diagonals, and the area of the square.

8. Polar coördinates. The position of a point P in a plane is also determined by its distance r, in terms of a given unit of distance, from a fixed point O, called the **origin** or **pole**, and the angle θ which OP makes with the positive direction of a fixed linear scale OX, called the **initial line** or **axis** (Fig. 8). OP is called the **radius vector**, and the angle XOP the **vectorial angle**, or simply the angle. r and θ are the **polar coördinates** of P. The notation $P \equiv (r, \theta)$, or $P(r, \theta)$, means that r and θ are the polar coördinates of P.

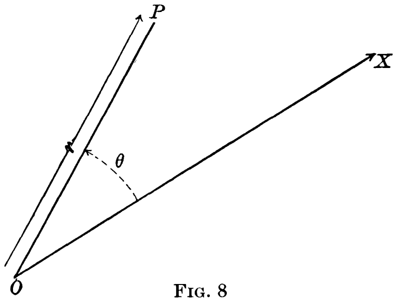


FIG. 8

A given number r and a given angle θ determine uniquely the position of a point in a plane with reference to a fixed origin and initial line. For, imagine the initial line OX (Fig. 9) to be rotated through the given angle θ about O into the position OX'. On OX' mark the point which represents the given number r. There is but one such point. For example, the point $P(-5, -30^\circ)$ is obtained by rotating OX through the angle -30° and marking the point 5 units from O on the negative end of the scale OX'.

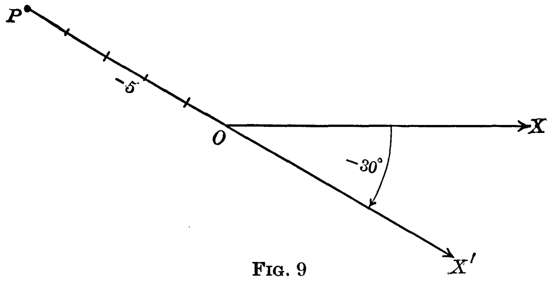


FIG. 9

On the other hand, a given point has many sets of polar coördinates. For example, the point P in Fig. 9 is $(-5, -30^\circ)$, $(5, 150^\circ)$, $(-5, 330^\circ)$, $(5, -210^\circ)$, etc. It is always possible, however, and usually most convenient, to choose the polar coördinates of a point so that the radius vector shall be a positive number, and $0 < \theta \leqq 2\pi$.

9. Relation between rectangular coördinates and polar coördinates. In Fig. 10, let O be the origin and OX the initial line, so that the polar coördinates of any point as P are:

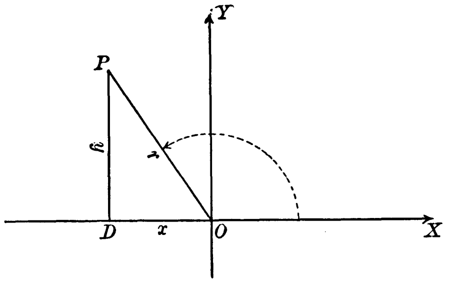


FIG. 10

$$r = OP \text{ and } \theta = XOP.$$

Let OX and OY be the X- and Y-axes, respectively, so that the rectangular coördinates of P are

$$x = OD \text{ and } y = DP.$$

Now, wherever the point P may be located in the plane, we always have

$$x = r \cos \theta \text{ and } y = r \sin \theta. \tag{1}$$

From equations (1), by squaring and adding, we obtain

$$x^2 + y^2 = r^2. \tag{2}$$

Also from equations (1), we have

$$\theta = \text{arc}\cos\frac{x}{r} = \text{arc}\sin\frac{y}{r}. \tag{3}$$

From (2) and (3),

$$r = \pm\sqrt{x^2+y^2} \text{ and } \theta = \text{arc}\cos\frac{x}{\pm\sqrt{x^2+y^2}} = \text{arc}\sin\frac{y}{\pm\sqrt{x^2+y^2}}$$

$$= \text{arc}\tan\frac{y}{x}. \tag{4}$$

Equations (1) serve to change the polar coördinates of a point into rectangular coördinates; and equations (4) are used to change rectangular coördinates into polar coördinates. For example, the rectangular coördinates of the point $P(-5,\ -30°)$ are

$$x = -5\cos(-30°) = \frac{-5\sqrt{3}}{2},\ y = -5\sin(-30°) = \frac{5}{2}. \quad \text{(See Fig. 9.)}$$

Again, the polar coördinates of the point $P(-3,\ -4)$ are

$$r = \sqrt{9+16} = 5,\ \theta = \text{arc}\cos(-\tfrac{3}{5}) = \text{arc}\sin(-\tfrac{4}{5}) = 233°\ 8'.$$

In solving this problem, r was taken to be the positive square root of 25. With $r = -5$,

$$\theta = \text{arc}\cos(\tfrac{3}{5}) = \text{arc}\sin(\tfrac{4}{5}) = 53°\ 8'.$$

EXERCISES

1. Plot the points $(3,\ -30°)$, $\left(-4,\ \frac{2\pi}{3}\right)$, (3, 2 radians). Find the rectangular coördinates of these points.

2. Find the polar coördinates of the points whose rectangular coördinates are $(3,\ -7)$, $(4,\ 2)$, $(-3,\ -5)$. Plot the points.

3. Where do the points lie for which the radius vector is constant? For which the vectorial angle is constant?

4. If $x = 4$ and $r = 5$, find y and θ. Is there more than one point satisfying the given conditions?

5. With a centigrade scale on the X-axis and a Fahrenheit scale on the Y-axis, plot a number of points whose coördinates represent the same absolute temperature. For example (0, 32), (5, 41), etc. Try to show that all these points must lie on a straight line, and to find where this straight line meets the X-axis.

6. Plot a number of points for which the radius vector is twice the abscissa. Join the points plotted. Do they lie on a straight line?

7. Elevations of points on the ground above a fixed datum plane are sometimes expressed in meters, while the distances of these points from a given place of beginning may be expressed in feet.

Plot the points whose elevations and distances are as follows:

DISTANCE	ELEVATION
100 feet	3.2 meters
200 feet	6.0 meters
250 feet	8.0 meters
300 feet	7.0 meters
400 feet	5.0 meters
500 feet	3.0 meters
600 feet	−2.0 meters

A broken line drawn through the points thus determined is called a **profile**.

8. Reduce the elevations of exercise 7 to feet and plot the same profile.

9. From a point O, the azimuths and distances to three points A, B, and C are as follows:

	AZIMUTH	DISTANCE
A	120°	10 rds.
B	180°	15 rds.
C	240°	12 rds.

Make an accurate map of the triangle ABC. With O as origin and OB as Y-axis, compute the rectangular coördinates of A, B, and C. With O as pole and OB as initial line, compute the polar coördinates of A, B, and C.

10. Construct a scale on the X-axis, the unit of measure representing one foot; and a scale on the Y-axis, the unit of measure representing one meter. Take one meter equivalent to 3.28 feet. Plot a number of points whose coördinates represent the same distance, for example (3.28, 1), (6.56, 2), etc. Show that these points lie on a straight line passing through the origin.

11. Construct a scale on the X-axis representing British money, and a scale on the Y-axis representing American money. Take £1 equivalent to \$4.87. Plot a number of points whose coördinates represent the same value, for example (1, 4.87), (2, 9.74), etc. Show that these points lie on a straight line passing through the origin.

12. Plot the points whose polar coördinates are $(-6, 20°)$, $(-5, -315°)$, $\left(-4, \frac{\pi}{2}\right)$, $\left(-3, \frac{3\pi}{2}\right)$. Change the coördinates of these points so that r and θ shall be positive, and θ less than 360°.

13. Change the polar coördinates of the points in exercise 12 to rectangular coördinates.

14. With a convenient unit, mark the points U and B on the X-axis, representing the numbers 1 and b, respectively. On the Y-axis, mark a point A, representing the number a. Join A to U, and through B draw a parallel to AU, meeting the Y-axis in C. Prove that C represents the number ab.

15. With a convenient unit, mark the point U, on the X-axis, representing the number 1; and on the Y-axis the points B and A, representing the numbers b and a, respectively. Join B to U, and through A draw a parallel to BU, meeting the X-axis in the point C. Show that C represents the number $\frac{a}{b}$.

16. With a convenient unit, mark the points U and A on the X-axis, representing the numbers -1 and a, respectively (a being a positive number). On UA as diameter, draw a circle and prove that it meets the Y-axis in points representing the numbers $\pm\sqrt{a}$. In this way construct geometrically $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$.

CHAPTER II

DIRECTED SEGMENTS AND AREAS OF PLANE FIGURES

10. Projections upon the coördinate axes. Let P_1P_2 be any directed segment. Through $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ draw parallels to the axes as shown in Fig. 11. The segments D_1D_2 and E_1E_2, thus determined upon the axes, are called the **projections** of P_1P_2 upon the X-axis and upon the Y-axis, respectively. The projections themselves are directed segments, and therefore (Art. 3)

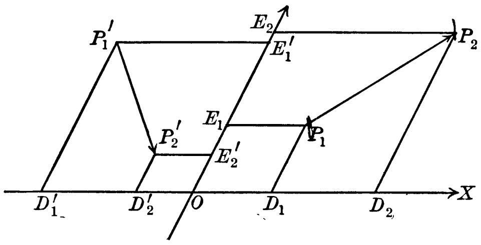


FIG. 11

$$\begin{aligned} D_1D_2 &= D_1O + OD_2 = -OD_1 + OD_2 = x_2 - x_1, \\ E_1E_2 &= E_1O + OE_2 = -OE_1 + OE_2 = y_2 - y_1. \end{aligned} \qquad (1)$$

If we call P_1 the initial point and P_2 the terminal point, the projection of P_1P_2 upon the X-axis is found by subtracting the x-coördinate of its initial point from the x-coördinate of its terminal point. Similarly, the projection upon the Y-axis is found by subtracting the y-coördinate of the initial point from the y-coördinate of the terminal point.

11. Inclination and slope of a directed segment. Let the coördinate axes be rectangular. Through the initial point of a directed segment draw a line parallel to the X-axis, having its positive direction the same as that axis. The line P_1D (Fig. 12) is this parallel. The positive angle through which it is necessary to rotate this parallel to make it coincide with the given directed segment is the **inclination** of the segment. Thus the angle DP_1P_2

is the inclination of each of the directed segments in Fig. 12 The inclination may have any value from 0° to 360° inclusive.

The tangent of the inclination is called the **slope** of the directed segment. Through P_2, the terminal point of the segment, draw a line parallel to the Y-axis and let it meet the parallel to the X-axis in P_3. The tangent of the angle DP_1P_2 is then $\frac{P_3P_2}{P_1P_3}$. But P_3P_2 is equivalent to the projection of P_1P_2 upon the Y-axis and P_1P_3 is equivalent to the projection upon the X-axis. Hence,

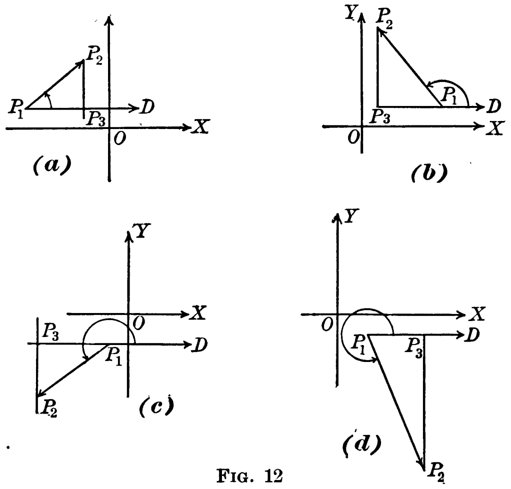


FIG. 12

$$\textbf{slope of } P_1P_2 = \textbf{tan } DP_1P_2 = \frac{y_2 - y_1}{x_2 - x_1} \qquad (2)$$

For example, the slope of the segment joining $(-4, -2)$ to $(2, 5)$ is

$$\frac{5-(-2)}{2-(-4)} = \frac{7}{6}.$$

Although reversing the direction of a segment changes its inclination by 180°, it does not change its slope. For,

$$\text{slope of } P_2P_1 = \frac{y_1 - y_2}{x_1 - x_2} = \text{slope of } P_1P_2.$$

EXERCISES

1. Determine the projections, the inclination, and the slope of each of the following directed segments:

(*a*) $(-2, 4)$, $(3, 6)$; (*b*) $(-5, 7)$, $(-4, -2)$; (*c*) $(3, -2)$, $(5, 6)$; (*d*) $(-3, 2)$, $(-2, -3)$.

Draw each segment.

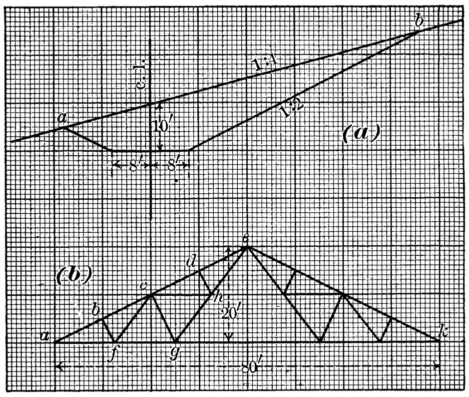
2. If the coördinate axes make an angle of 60° with each other, determine the angle which the directed segment (2, 1), (4, 2) makes with each axis.

3. Draw the triangle whose vertices are (1, 2), (5, 4), (2, 6), using rectangular coördinates.

(*a*) Find the lengths of the projections of the sides upon the X-axis. What is the sum of these projections?

(*b*) Find the inclination of each side. How can the angles of the triangle be found from these inclinations?

4. Show that the sum of the projections of the sides of any triangle upon either axis is zero, provided that the sides be taken in order around the triangle.



5. Fig. *a* represents a railroad cutting in a side-hill. The slope of the natural surface is 1 : 4 and that of the proposed cutting is 1 : 2. At what heights above the bottom of the cut and at what distances out from the center line are the points of intersection *a* and *b*?

6. Fig. *b* is the outline of a roof truss of 80-ft. span and 20-ft. rise. The spaces *ab*, *bc*, etc., are equal and the members *bf*, *cg*, and *dh* are perpendicular to the member *ae*. Calculate the slopes of *ae*, *bf*, *fc*, and *ge* with respect to a horizontal axis *ak*.

7. Calculate the slopes of *cf* and *ch* with respect to the line *ae* taken as the horizontal axis.

12. The length of a segment. The problem to find the distance between two points whose coördinates are given, that is, the length of the segment joining them, depends upon the problem of finding the length of one side of a triangle when the other two sides and their included angle are given. Thus, with cartesian coördinates, let

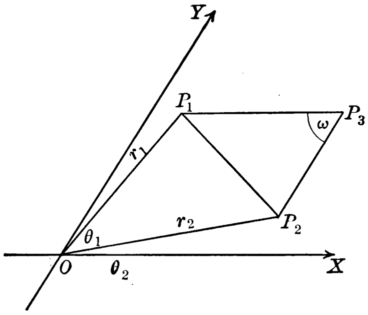


FIG. 13

$P_1(x_1, y_1)$, $P_2(x_2, y_2)$ be the given points, and let the angle XOY be ω (Fig. 13). Draw parallels to the axes through P_1 and P_2 forming the triangle $P_1P_3P_2$. The sides P_1P_3 and P_2P_3 are known from the given coördinates of P_1 and P_2, and the angle $P_1P_3P_2 =$ the angle $XOY = \omega$. Therefore, by the law of cosines,

$$\overline{P_1P_2}^2 = \overline{P_1P_3}^2 + \overline{P_2P_3}^2 - 2\,\overline{P_1P_3}\cdot\overline{P_2P_3}\cos\omega. \tag{1}$$

If P_1P_2 is a directed segment, P_1P_3 and P_3P_2 are respectively equivalent to its projections upon the X- and Y-axes. In terms of these projections, formula (1) becomes

$$\overline{P_1P_2}^2 = \overline{P_1P_3}^2 + \overline{P_3P_2}^2 + 2\,\overline{P_1P_3}\cdot\overline{P_3P_2}\cos\omega, \tag{2}$$

or (Art. 10),

$$\overline{P_1P_2}^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + 2(x_2 - x_1)(y_2 - y_1)\cos\omega. \tag{3}$$

With rectangular coördinates, $\omega = 90°$ and the triangle $P_1P_3P_2$ is right-angled at P_3. We have, then, only to find the length of the hypotenuse, having given the other two sides.

With polar coördinates, let $P_1 \equiv (r_1, \theta_1)$ and $P_2 \equiv (r_2, \theta_2)$. In the triangle P_1OP_2, two sides and the included angle are known, hence

$$\overline{P_1P_2}^2 = r_1^2 + r_2^2 - 2\,r_1r_2\cos(\theta_2 - \theta_1). \tag{4}$$

EXERCISES

1. The angle between the axes being 45°, find the distance between the points $(-3, -5)$ and $(5, 2)$.

2. Plot the points whose polar coördinates are $\left(-3, \frac{\pi}{6}\right)$ and $\left(2, \frac{2\pi}{3}\right)$ and find the distance between them.

3. The rectangular coördinates of P_1 are $(3, -2)$ and the polar coördinates of P_2 are $(-5, 60°)$. Find the length of P_1P_2.

4. The vertices of a triangle are situated at the points $(5, -2)$, $(-4, 7)$, and $(7, -3)$, in rectangular coördinates. Find the lengths of the sides.

5. Milwaukee is 80 miles east of Madison and 80 miles north of Chicago. What are the polar coördinates of Chicago with respect to Madison as origin and the line from Madison to Milwaukee as axis? The polar coördinates of Portage being $\left(40, \frac{\pi}{2}\right)$, find the distance from Chicago to Portage.

6. Show that the formula (3) Art. 12, holds for all positions of the points P_1 and P_2.

13. Angle which one segment makes with another. Let the segments P_1P_2 and Q_1Q_2, produced if necessary, meet in the point A (Fig. 14). The angle which Q_1Q_2 makes with P_1P_2 is defined as *the positive angle through which it is necessary to rotate P_1P_2 about A until it coincides in direction with Q_1Q_2*. In the figure, this angle is P_2AQ_2 which is clearly the difference between the inclinations of the two segments. If θ_2 and θ_1 are respectively the inclinations of Q_1Q_2 and P_1P_2 and ϕ is the angle P_2AQ_2 then

$$\phi = \theta_2 - \theta_1. \qquad (1)$$

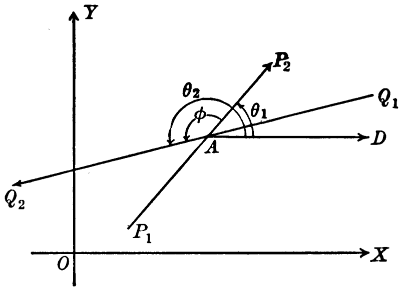


Fig. 14

Formula (1) holds only when $\theta_2 > \theta_1$. When $\theta_2 < \theta_1$, the angle which Q_1Q_2 makes with P_1P_2 is given by the formula

$$\phi = 2\,\pi + (\theta_2 - \theta_1), \qquad (2)$$

as the student may easily verify. In either case

$$\tan \phi = \tan\,(\theta_2 - \theta_1) = \frac{\tan \theta_2 - \tan \theta_1}{1 + \tan \theta_2 \tan \theta_1}. \qquad (3)$$

If m_2 and m_1 are respectively the slopes of Q_1Q_2 and P_1P_2, formula (3) becomes

$$\mathbf{\tan \phi = \frac{m_2 - m_1}{1 + m_2 m_1}}. \qquad (4)$$

As an example of the use of formula (4), we will find the angle which the segment joining (3, 5) to (– 2, – 6) makes with the segment joining (– 1, 2) to (3, – 4) (Fig. 15). Here m_2, the slope of Q_1Q_2, is equal to $-\frac{3}{2}$ and m_1, the slope of P_1P_2, is equal to $\frac{11}{5}$. Hence

$$\tan \phi = \frac{-\frac{3}{2} - \frac{11}{5}}{1 - \frac{33}{10}} = \frac{37}{23}, \text{ and } \phi = 58° \, 8'.$$

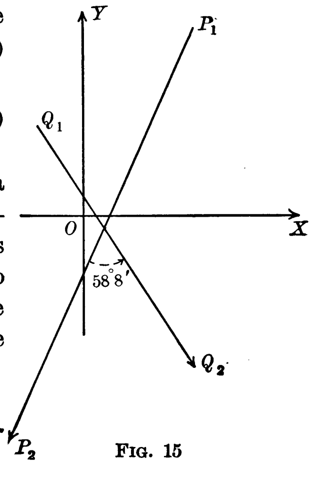


Fig. 15

14. Parallel segments. Parallel segments either have equal inclinations, as at (*a*) (Fig. 16), or else their inclinations differ by 180° as at (*b*). In either case, their slopes are the same (Art. 11). For example, the segment joining (1, 2) to (−2, −3) is parallel to the segment joining (2, −1) to (5, 4), since the slope of each is $\frac{5}{3}$.

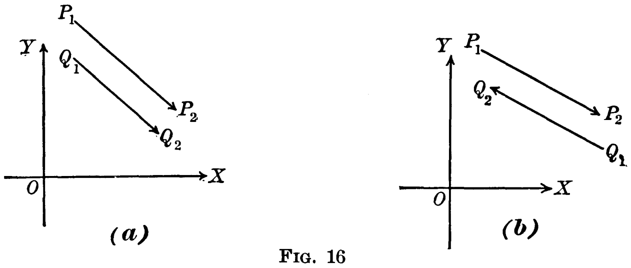


FIG. 16

15. Perpendicular segments. When two segments are perpendicular to each other, their inclinations differ by an odd multiple of 90° and therefore, in every case,

$$\tan\theta_2 = -\cot\theta_1 = -\frac{1}{\tan\theta_1}, \text{ or}$$

$$\boldsymbol{m_2 = -\frac{1}{m_1}}. \tag{1}$$

Thus, the slope of each segment is the negative reciprocal of the slope of the other.

Conversely, *if the product of the slopes of two segments is* −1, *the segments are perpendicular to each other.* For then the tangent of the inclination of one of them is equal to the negative of the cotangent of the inclination of the other. Hence, their inclinations differ by an odd multiple of 90°; that is, the segments are perpendicular to each other.

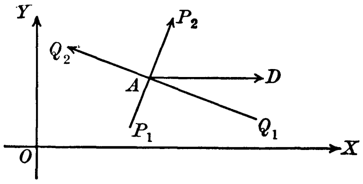


FIG. 17

In Fig. 17, P_1P_2 makes an angle of 90° with Q_1Q_2, but Q_1Q_2 makes an angle of 270° with P_1P_2.

EXERCISES

1. Find the angle which the segment (– 3, 2), (4, – 1) makes with the segment (–3, 2), (8, 5). Draw the figure.

2. Compute the lengths of the sides and the angles of the triangle whose vertices are (– 3, 2), (4, – 1), and (8, 5). Draw the figure.

3. Show that the triangle whose vertices are (3, 2), (0, 3.5), and (1, 5.5) is right-angled.

4. Show that the segments (– 3, 5), (3, 2) and (– 1, 6), (3, 4) are parallel. Draw the figure and compute the perpendicular distance between the segments.

5. Join the extremities of the segments in the preceding exercise and compute the area of the quadrilateral so formed.

6. Draw the diagonals of the quadrilateral in the preceding exercise and find the acute angle which one makes with the other.

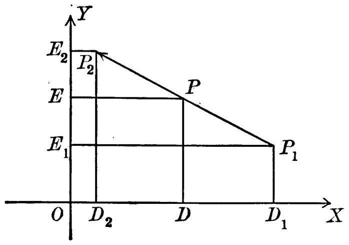


Fig. 18

16. Point bisecting a given segment. Let $P_1 \equiv (x_1, y_1)$ and $P_2 \equiv (x_2, y_2)$; it is required to find the coördinates of the point $P \equiv (x, y)$ bisecting the segment P_1P_2 (Fig. 18). The parallels to the axes through P must bisect the projections D_1D_2 and E_1E_2 in D and E, respectively. Hence (Art. 10)

$$x = OD = OD_1 + D_1D = x_1 + \frac{x_2 - x_1}{2},$$

and
$$y = OE = OE_1 + E_1E = y_1 + \frac{y_2 - y_1}{2},$$

whence
$$x = \frac{x_1 + x_2}{2}, \quad y = \frac{y_1 + y_2}{2}. \tag{1}$$

For example, the coördinates of the point bisecting the segment (1, 3), (– 3, – 1) are

$$x = \frac{1-3}{2} = -1,$$

$$y = \frac{3-1}{2} = 1.$$

17. Point dividing a given segment in a given ratio. The results of the preceding article can be generalized. Thus, suppose the point P (Fig. 19) divides the segment P_1P_2 so that

$$\frac{P_1P}{PP_2} = r.$$

Then the points D and E divide the projections of P_1P_2 in the same ratio. Hence,

$$\frac{P_1P}{PP_2} = \frac{D_1D}{DD_2} = \frac{(x - x_1)}{(x_2 - x)} = r,$$

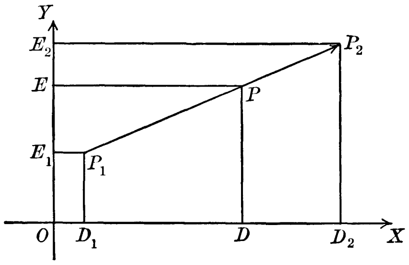


Fig. 19

and
$$\frac{P_1P}{PP_2} = \frac{E_1E}{EE_2} = \frac{(y - y_1)}{(y_2 - y)} = r.$$

Solving these equations for x and y, we have

$$\begin{aligned} x &= \frac{x_1 + rx_2}{r + 1}, \\ y &= \frac{y_1 + ry_2}{r + 1}. \end{aligned} \tag{1}$$

For example, to find the coördinates of the point dividing the segment $P_1 \equiv (2, 4)$, $P_2 \equiv (-3, -2)$ in the ratio 2 : 3, we have $r = \frac{2}{3}$. Substituting in the above formulas we find $x = 0$ and $y = \frac{8}{5}$. Hence the point $(0, \frac{8}{5})$ divides the given segment in the ratio 2 : 3.

EXERCISES

1. Find the coördinates of the points which bisect the sides of the triangle (2, 5), (– 2, 2), (4, – 5).

2. In the preceding exercise, join the vertices to the mid-points of the sides opposite and show that the points dividing each segment from vertex to opposite side in the ratio 2 : 1 coincide.

3. Generalize the preceding exercise and thus prove that the medians of any triangle meet in a point.

4. Show that the points (2, 3), (4, 1), (8, 2) and (6, 4) form a parallelogram. Find the coördinates of the mid-points of the diagonals.

5. Find the coördinates of the points which trisect the segment (6, 4), (– 3, 1).

6. Find the coördinates of the point P dividing the segment $P_1 \equiv (3, 4)$, $P_2 \equiv (-2, -6)$ in the ratio $3:5$. Prove the result by calculating the lengths of the segments P_1P, PP_2 and showing that their ratio is $\frac{3}{5}$.

7. The segment in the preceding exercise crosses both axes. Find the coördinates of the points of crossing.

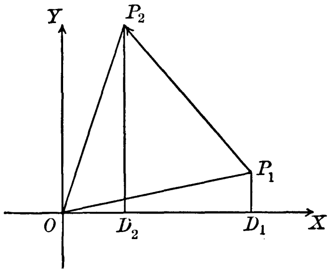


FIG. 20

18. Area of a triangle, one vertex at the origin. To find the area of the triangle OP_1P_2 (Fig. 20), let $P_1 \equiv (x_1, y_1)$, $P_2 \equiv (x_2, y_2)$, and D_1D_2, the projection of P_1P_2 upon the x-axis. Then, if Δ represents the required area,

$$\begin{aligned}\Delta &= \text{trapezoid } P_1P_2D_2D_1 + \text{triangle } P_2OD_2 - \text{triangle } OD_1P_1\\ &= \frac{(y_1+y_2)(x_1-x_2)}{2} + \frac{x_2y_2}{2} - \frac{x_1y_1}{2}.\end{aligned}$$

Hence, we have

$$\Delta = \frac{(x_1y_2 - x_2y_1)}{2}. \qquad (1)$$

The expression $x_1y_2 - x_2y_1$ is a *determinant* and is often written thus;

$$\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}.$$

In determinant notation, the formula for the area of the triangle is then

$$\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}. \qquad (2)$$

For example, the area of the triangle formed by joining the extremities of the segment $P_1 \equiv (3, 1)$, $P_2 \equiv (1, 3)$ to the origin is

$$\frac{1}{2}\begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix} = 4.$$

19. Sign of the expression $(x_1y_2 - x_2y_1)$. The sign of the expression $(x_1y_2 - x_2y_1)$ is not the same for all positions of the segment P_1P_2. Thus, if $P_1 \equiv (3, 1)$ and $P_2 \equiv (1, 3)$, the expression has the value $+8$, while for the segment $P_1 \equiv (1, -2)$,

$P_2 \equiv (-1, 1)$, which has the same length and the same slope as the former, the expression $(x_1y_2 - x_2y_1)$ has the value -1.

Changing to polar coördinates by means of the relations

$$x_1 = r_1 \cos \theta_1,\ y_1 = r_1 \sin \theta_1,\ x_2 = r_2 \cos \theta_2,\ y_2 = r_2 \sin \theta_2 \text{ (Art. 9)},$$

the expression $(x_1y_2 - x_2y_1)$ becomes

$$r_1r_2(\cos \theta_1 \sin \theta_2 - \cos \theta_2 \sin \theta_1) = r_1r_2 \sin (\theta_2 - \theta_1).$$

Since r_1 and r_2 may be considered as positive numbers (Art. 8), the sign of $(x_1y_2 - x_2y_1)$ will be positive when $\sin (\theta_2 - \theta_1)$ is positive; that is, when $\theta_2 - \theta_1$ is an angle in the first, or the second, quadrant. In either case, the segment P_1P_2 has a position such that, in passing from P_1 to P_2, the origin lies to the left as at (*a*), Fig 21.

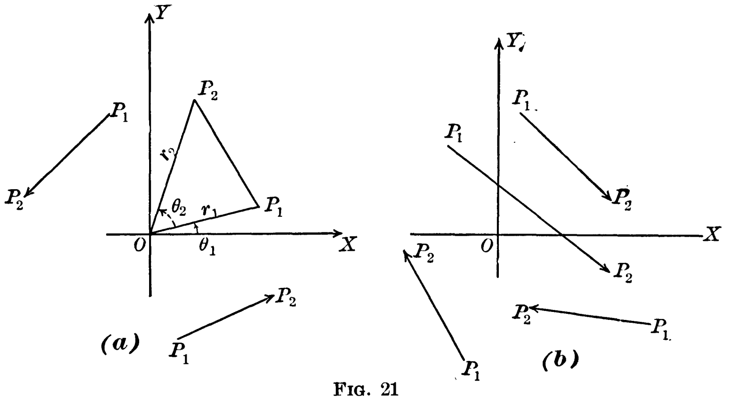


FIG. 21

On the other hand, the expression $(x_1y_2 - x_2y_1)$ will be negative when $\theta_2 - \theta_1$ is an angle in the third, or the fourth, quadrant; and then the segment P_1P_2 has a position such that, in passing from P_1 to P_2, the origin lies to the right as at (*b*).

Conversely, if the segment P_1P_2 has a position such that the origin lies to the left (or the right) when the segment is traversed from P_1 to P_2, the sign of $(x_1y_2 - x_2y_1)$ will be positive (or negative). For then the angle $\theta_2 - \theta_1$ must lie in the first, or the second, quadrant (or in the third, or the fourth, quadrant). Consequently the area of the triangle OP_1P_2, which is $\frac{1}{2}(x_1y_2 - x_2y_1)$, is positive when the origin lies to the left, as at (*a*), Fig. 21, and negative when the origin lies to the right, as at (*b*).

EXERCISES

1. $P_1 \equiv (5, 3)$ and $P_2 \equiv (-1, -3)$; determine the area of OP_1P_2, O being the origin. Explain the sign of the result. Draw the figure.

2. If O is the pole, show that the area of the triangle OP_1P_2 is

$$\tfrac{1}{2} r_1r_2 \sin(\theta_2 - \theta_1),$$

where $P_1 \equiv (r_1, \theta_1)$ and $P_2 \equiv (r_2, \theta_2)$.

3. If $P_1 \equiv \left(5, \frac{\pi}{3}\right)$ and $P_2 \equiv (3, -30°)$, find the area of OP_1P_2.

4. Given $P_1 \equiv (3, -60°)$ and $P_2 \equiv (3, 4)$, find the area of OP_1P_2.

5. When the segment P_1P_2 passes through the origin, what is the value of the expression $(x_1y_2 - x_2y_1)$?

6. If $P_1 \equiv (-3, 1)$ and $P_2 \equiv (1, -2)$, in which quadrant is the angle $\theta_2 - \theta_1$? Draw the figure and find the area of OP_1P_2.

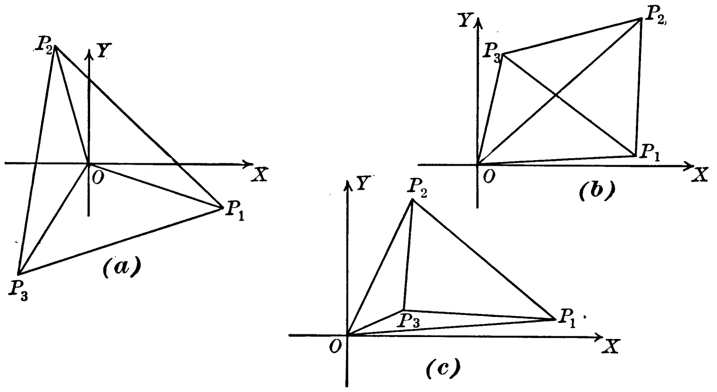


Fig. 22

20. Area of a triangle, vertices in any position. Join the vertices of the triangle to the origin O. Let P_1, P_2, P_3 (Fig. 22) be the vertices, taken in counterclockwise order about the triangle. The area of $P_1P_2P_3$ is then given by the formula

$$\textbf{area } P_1P_2P_3 = \textbf{area } OP_1P_2 + \textbf{area } OP_2P_3 + \textbf{area } OP_3P_1. \quad (1)$$

Thus in (*a*), each of the component triangles has a positive area, by the preceding article, and their sum is obviously the area of $P_1P_2P_3$. In (*b*), the areas of OP_1P_2 and OP_2P_3 are positive numbers, while the area of OP_3P_1 is a negative number. The algebraic sum of these numbers is clearly the area of $P_1P_2P_3$. Finally, in (*c*), the area of OP_1P_2 is a positive number, while the areas of the remaining two triangles are expressed by negative numbers. The algebraic sum of these numbers is again the area of $P_1P_2P_3$.

Replacing the area of each component triangle in (1) by its value in terms of the coördinates of the vertices (Art. 18), we have

$$\textbf{area } \boldsymbol{P_1P_2P_3} = \tfrac{1}{2}[(x_1y_2 - x_2y_1) + (x_2y_3 - x_3y_2) + (x_3y_1 - x_1y_3)]. \quad (2)$$

The area can be expressed in determinant notation. Thus

$$\textbf{area } \boldsymbol{P_1P_2P_3} = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}, \quad (3)$$

since, if the determinant is expanded, the result agrees with formula (2).

The following is a convenient rule for computing the area.

Let $P_1P_2P_3$ be the vertices, taken in counterclockwise order about the triangle. Arrange the coördinates in rows, thus

$$\begin{matrix} x_1 & x_2 & x_3 & x_1 \\ y_1 & y_2 & y_3 & y_1 \end{matrix};$$

multiply each x by the y standing in the next column to the right and add the products, thus

$$x_1y_2 + x_2y_3 + x_3y_1;$$

multiply each y by the x in the next column to the right and add the products, thus

$$y_1x_2 + y_2x_3 + y_3x_1;$$

subtract the latter sum from the former and take half the difference, the result is the area of the triangle $P_1P_2P_3$.

For example, to find the area of the triangle whose vertices are $P_1 \equiv (-1, 3)$, $P_2 \equiv (3, 2)$, $P_3 \equiv (5, 4)$ (Fig. 23), arrange the coördinates as

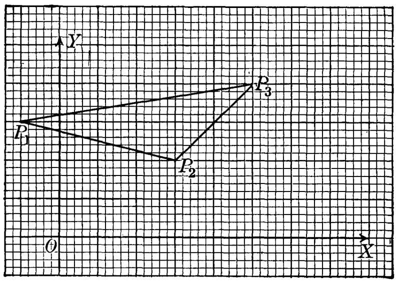


FIG. 23

in the rule, being careful to note that the vertices are taken in counterclockwise order; thus

$$\begin{array}{cccc} -1 & 3 & 5 & -1 \\ 3 & 2 & 4 & 3. \end{array}$$

The area is then

$$\tfrac{1}{2}[(-2+12+15) - (9+10-4)] = 5.$$

If the vertices are taken in clockwise order about the triangle, the result obtained by using formula (2) or formula (3) or the rule just stated will be numerically the same but will be negative in sign, as the student may easily verify.

EXERCISES

1. Find the area of the triangle whose vertices are $(2, -6)$, $(-9, 7)$, $(8, 3)$.

2. Find the area of the triangle whose vertices are $(-1, -2)$, $(2, 1)$, and $(3, 2)$. Explain the result.

3. Find the area of the triangle whose vertices in polar coördinates are $\left(6, \frac{\pi}{3}\right)$, $\left(-6, \frac{2\pi}{3}\right)$, and $\left(2, \frac{3\pi}{4}\right)$. Draw the figure.

4. The vertices of a quadrilateral are $(-1, 6)$, $(8, 10)$, $(10, -2)$, and $(-5, -8)$. Compute the area of the quadrilateral by dividing it into two triangles. Draw the figure.

5. When three points are in the same straight line, they are said to be **collinear**. Show that the points $(1, 3)$, $(3, 1)$, and $(4, 0)$ are collinear.

6. Where will the line joining the points $(2, 5)$ and $(3, 6)$ meet the axes?

7. If $P_1(x_1, y_1)$, $P_2(x_2, y_2)$, and $P_3(x_3, y_3)$ are three collinear points, show that

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_1 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0.$$

State and prove the converse.

21. Area of any polygon. Formula (2) of article 20 can be extended to find the area of any polygon when the coördinates of the vertices are given. Thus when the vertices, taken in counterclockwise order about the polygon, are joined to the origin, a

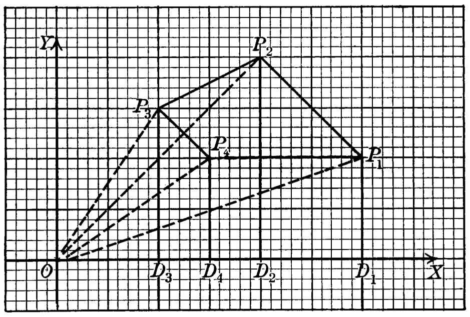


FIG. 24

number of component triangles are formed (Fig. 24). It is geometrically evident that the algebraic sum of the areas of these triangles is the area of the polygon. A convenient rule for computing the area of a polygon is, therefore, obtained by extending the rule in Art. 20. Thus, write the x's over the y's and form the cross-products:

$$\begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 \cdots & x_n & x_1, \\ y_1 & y_2 & y_3 & y_4 \cdots & y_n & y_1. \end{array}$$

The required area is then

$$\Delta = \tfrac{1}{2}\left[(x_1y_2 + x_2y_3 + x_3y_4 + \cdots + x_ny_1) - (y_1x_2 + y_2x_3 + y_3x_4 + \cdots + y_nx_1)\right]. \qquad (1)$$

For example, to find the area of the quadrilateral whose vertices are, in counterclockwise order (8, 10), (− 1, 6), (− 5, − 8), and (10, − 2), we have

$$\begin{array}{ccccc} 8 & -1 & -5 & 10 & 8 \\ 10 & 6 & -8 & -2 & 10 \end{array}$$

and the area is, therefore,

$$\tfrac{1}{2}[(48 + 8 + 10 + 100) - (-10 - 30 - 80 - 16)] = 151.$$

EXERCISES

1. The vertices of a hexagon are (6, 1), (3, − 10), (− 3, − 5), (− 12, 0), (− 4, 6), and (9, − 4). Draw the hexagon and compute its area.

2. A surveyor finds that the corners of a four-sided field are situated, with respect to a north and south road and an east and west road, as follows: $A \equiv (25, 32)$, $B \equiv (48, 65)$, $C \equiv (94, -10)$, and $D \equiv (30, -40)$. Distances are measured in rods. Make a map of the field and compute the number of acres it contains. (160 square rods = 1 acre).

3. From a point O in a quadrangular field, the distances and directions to the corners are as follows: $A \equiv 120$ feet, N. 65° E.; $B \equiv 216$ feet, N. 32° W. $C \equiv 320$ feet, S. 74° W.; $D \equiv 65$ feet, S. 23° E. Make a map of the field and compute its area.

4. In surveying, points are frequently located by azimuth and distance from a given point, that is, by polar coördinates. It frequently becomes necessary to plot the outlines of tracts of land determined in this way and to calculate areas.

Draw the polygonal figure whose vertices are determined by the following azimuths and distances:

AZIMUTH	DISTANCE	AZIMUTH	DISTANCE
125°	115 feet	342°	175 feet
170°	160 feet	15°	40 feet
250°	200 feet	73°	10 feet

(*a*) Calculate the coördinates of the vertices of this figure referred to N. and S. and E. and W. lines through the given fixed point, as origin.

(*b*) Calculate the directions of the sides of this figure.

(*c*) Calculate the area of the polygon.

5. Fig. 24 A represents a cross section of one side of a railroad cutting. Calculate the area of this section, using coördinates as shown.

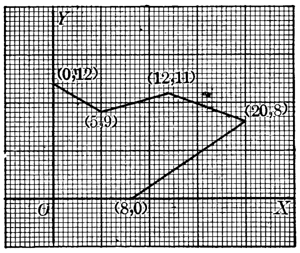


FIG. 24 A

In railroad field books the data for this problem would generally be recorded as follows:

Center

$$\left\{ \frac{12}{0} \times \frac{9}{5} \times \frac{11}{12} \times \frac{8}{20} \times \frac{0}{8} \right\}$$

The ordinate, or depth of cutting, is written above and the distance out from the center line (abscissa) is written below. The coördinates $(\frac{0}{8})$ are not actually recorded as the number 8 is the fixed width of the bottom of the cut. Arranging the coördinates in the above manner, the correct result is obtained by taking positive products along diagonal lines sloping downwards towards the right (shown by full lines) and negative products along the other diagonals (shown by dotted lines).

6. Compute the area of cross section given by the following cross section notes, left and right of the center line:

Center

$$(\tfrac{0}{8})\ \tfrac{4}{14}\ \tfrac{6}{10}\ \tfrac{5}{5}\ \tfrac{10}{0}\ \tfrac{10}{5}\ \tfrac{12}{15}\ \tfrac{18}{25}\ \tfrac{16}{32}\ (\tfrac{0}{8})$$

What side slope of the finished cut has been assumed in this problem?

7. Drop perpendiculars from the vertices of a polygon upon the X-axis, as in Fig. 24. Show that the area of the polygon is the algebraic sum of the areas of the trapezoids thus formed. Compute the area of the hexagon in exercise 1 by this method.

CHAPTER III

FUNCTIONS AND THEIR GRAPHIC REPRESENTATION

22. Constants and variables. The numbers and magnitudes considered in mathematics are either constants or variables. The coördinates of a fixed point are constants; the coördinates of a moving point are variables.

23. Functions. *If to each given value of a variable x there correspond one or more values of a variable y, then y is called a function of x.*

As examples, the cost of a money order is a function of the amount; the temperature at a given place is a function of the time; the cost of insurance is a function of the age of the insured; the distance a body falls freely in space is a function of the time the body has been falling.

24. Notation. To denote that y is a function of x, the notation $y = f(x)$ (read y equals f of x) is used.

When several functions are to be considered in the same problem, different symbols are used. Thus, $y = f_1(x)$, $y = f_2(x)$, . . . (read y equals f_1 of x, y equals f_2 of x, . . .). Or use is made of Greek letters, as $y = \phi(x)$, $y = \psi(x)$, . . . (read y equals phi of x, y equals psi of x, . . .).

25. Determination of functional correspondence. A functional correspondence can be *established*, or *set up*, between two variables in different ways. Thus, the correspondence may be primarily established:

I. By an equation connecting the two variables, as $y = x^2$;

II. By a table exhibiting corresponding values of the variables, as a table of logarithms;

III. By a curve drawn automatically, thus exhibiting graphically the correspondence between two variables.

26. Dependent and independent variables. If functional correspondence is established by an equation, the value (or values) of the function can, in general, be readily computed for any value assigned to the variable x. Thus, for example, if $y = 2x^2$, the value of y is easily computed for any assigned value of x. In general, y (the function) is called the **dependent variable**, and x, the **independent variable.**

27. Graphic representation. A table of corresponding values of a function and the independent variable can be derived from the equation by assigning to the independent variable a series of values, arbitrarily chosen, and computing the corresponding values of the function. With these corresponding values of x and y as rectangular coördinates, a series of points can be constructed. The ordinates of these points, taken together, form a **graphic representation** of the function. For the functions considered in this book, a curve can be drawn through all the points constructed as above. This curve is called the **graph of the function.** Thus, from the equation $y = 2x^2$, we obtain the following table of corresponding values:

$$x = -3,\ -2,\ -1,\ 0,\ 1,\ 2,\ 3,\ 4,\ \ldots .$$
$$y = 18,\ 8,\ 2,\ 0,\ 2,\ 8,\ 18,\ 32,\ \ldots .$$

The process of constructing the points whose coördinates are given in the table and drawing the curve through them, is called **plotting**. Figure 25 shows the completed graph. In constructing this graph, the unit of the scale on the X-axis was taken four times as great as the unit on the Y-axis in order to represent more of the curve within a small compass (cf. Art. 5). The curve shows at a glance the change in value of the function for any given change in value of the independent variable x. For example, as x changes from -3 to $+3$, the point which represents it moves from D_1 to D_6. At the same time the function y first decreases from 18 to 0 and then increases from 0 to 18.

When a function ceases to decrease and begins to increase, or vice versa, it is said to have a **turning point.** Thus, the function $y = 2x^2$ has a turning point at the origin.

When a function has no turning points, it is called a **monotone function.**

It is important to know whether a function has turning points or not, and, if it has, to know for what values of the independent variable they exist. At a turning point, the function has a **maximum** or a **minimum** value.

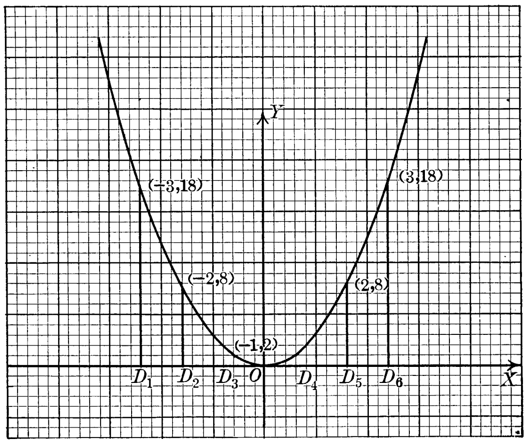


FIG. 25

EXERCISES

1. Draw the graph of the function $y = 2x + 3$. Find the coördinates of the points where the graph crosses the axes. Does the function have turning points, or is it a monotone function?

2. Draw the graph of the function expressing the law of falling bodies, $s = \frac{1}{2}gt^2$. Take $g = 32$ and corresponding values of s and t as ordinates and abscissas respectively. Make the unit of the scale on the t-axis ten times as great as the unit on the s-axis. Where is the turning point of the function s?

3. Draw the graph of the equation expressing Boyle's law, $pv = k$. Take $k = 4$ and corresponding values of p and v as ordinates and abscissas respectively. Make the units the same on each axis. Is p a monotone function of v or not?

4. Make careful drawings of the graphs of the function $y = x^n$ for $n = -1$, 0, 1, 2, and 3. Use the same axes and preserve the figures. For which of the given values of n is y not a monotone function of x?

5. Draw the graph of the function $y = 4 - x^2$. Determine the value of x for which this function has a turning point. Has the function a maximum or a minimum value at the turning point?

6. Draw the graph of the function $y = x^2 - 4x + 3$. Determine the coördinates of the turning point. Has the function a maximum or a minimum value at the turning point?

28. Single-valued and multiple-valued functions. When there corresponds but one value of the function to each given value of the independent variable, the function is called **single-valued.** If there is more than one value of the function corresponding to any given value of the variable, the function is called **multiple-valued.** For example, the function $y = 2x^2$ is single-valued; but the function $y^2 = 2x$ is multiple-valued, since to each value of x there correspond two values of y.

The following is a table of corresponding values for the function $y^2 = 2x$.

$x =$	-2,	-1,	0,	1,	2,	3,	4,	$\cdots$.
$y =$	Imag.,	Imag.,	0,	$\pm\sqrt{2}$,	± 2,	$\pm\sqrt{6}$,	$\pm\sqrt{8}$,	$\cdots$.

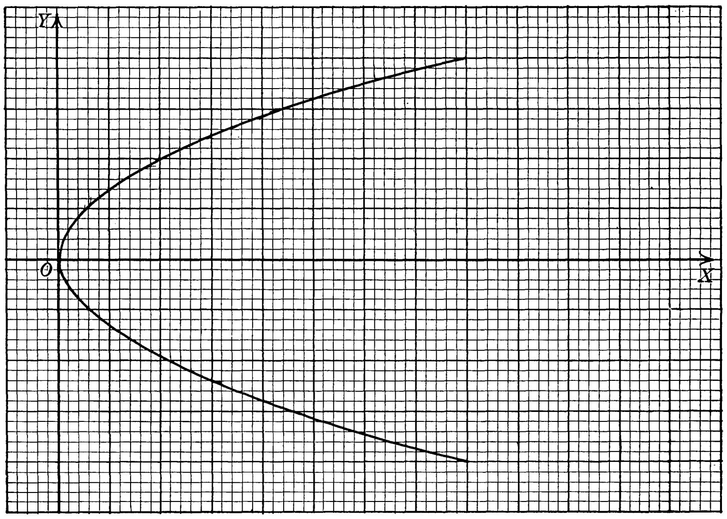


FIG. 26

The graph is shown in Fig. 26, where the same unit is used for the scale on the Y-axis as for the scale on the X-axis.

The curves in Figs. 25 and 26 are called parabolas.

29. Symmetry. *A curve, or graph, is symmetrical with respect to a straight line when the line bisects all the chords of the curve drawn perpendicular to it.*

For example, the parabola shown in Fig. 25 is symmetrical with respect to the Y-axis and the parabola in Fig. 26 is symmetrical with respect to the X-axis.

As another example, consider the single-valued function

$$y = 5x - 6 - x^2. \tag{1}$$

The following is a table of corresponding values:

$$\begin{array}{lrrrrrrrl} x = & 0, & 1, & 2, & \frac{5}{2}, & 3, & 4, & 5, & \cdots. \\ y = & -6, & -2, & 0, & \frac{1}{4}, & 0, & -2, & -6, & \cdots. \end{array}$$

The graph is shown in Fig. 27, where the units on the two axes are the same. We now see that the curve is symmetrical with

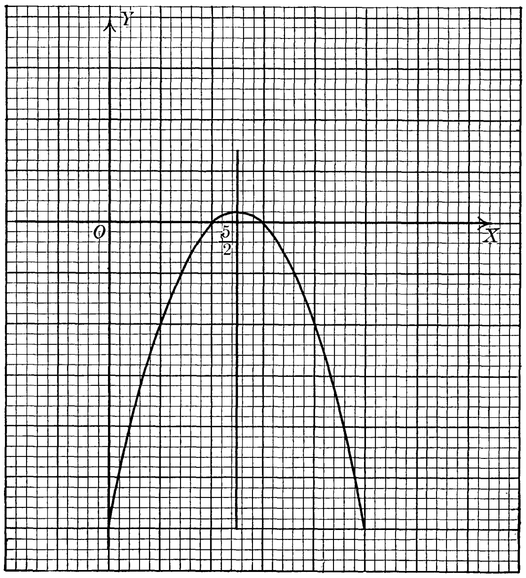


FIG. 27

respect to a line parallel to the Y-axis and passing through the point $(\frac{5}{2}, 0)$.

The symmetry is also shown, without plotting, by solving equation (1) for x. Thus,

$$x = \tfrac{5}{2} \pm \sqrt{\tfrac{1}{4} - y}, \tag{2}$$

and therefore, for a given value of y, x has two values represented by points equidistant from, and on opposite sides of, the point $x = \frac{5}{2}$. We thus see that the line parallel to the Y-axis through the point $x = \frac{5}{2}$ bisects the chords of the curve drawn perpendicular to it.

Notice that the function has a turning point at $x = \frac{5}{2}$, that the value of y is there equal to $\frac{1}{4}$, and that this is the maximum value of y.

A curve is symmetrical with respect to a point if the point bisects all the chords of the curve drawn through it.

For example, consider the function $y = x^3$. The graph is shown in Fig. 28, where the unit of the scale on the X-axis is taken five times as great as the unit of the scale on the Y-axis. We see that the origin bisects all the chords of the curve drawn through it. Hence the curve is symmetrical with respect to the origin.

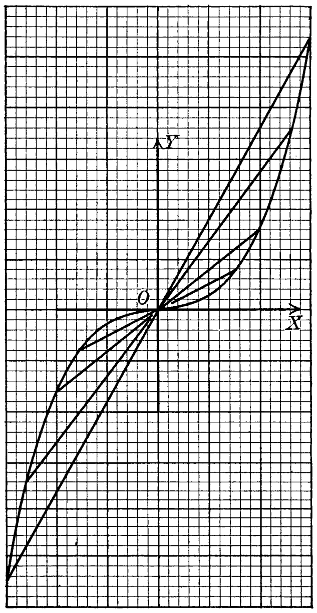


FIG. 28

30. Intercepts. The distances from the origin to the points where a graph crosses the axes are called the **intercepts**. Thus, in Fig. 27, the curve crosses the X-axis twice, at two and three units to the right of the origin. The X-intercepts are $+ 2$ and $+ 3$. The curve crosses the Y-axis six units below the origin. The Y-intercept is $- 6$.

The X-intercepts are the roots of the equation of the graph when y is put equal to zero, and the Y-intercepts are the roots of the equation when x is put equal to zero.

EXERCISES

1. Draw the graph of the function $y = x^2 - 2x - 3$. Find the position of the line of symmetry, the intercepts, and the coördinates of the turning point.

2. Draw the graph of the function y, when $y^2 - 2y = 2x - 1$. Find the position of the line of symmetry and the intercepts. Is y a single-valued, or multiple-valued function of x?

3. Given $yx = 4$. Show that the line bisecting the first and third quadrants is a line of symmetry. Find the coördinates of the points where this line meets the curve. Is y a monotone function of x? A single-valued function of x?

4. Show that the graph of $y = (x - 1)^3 + 2$ is symmetrical with respect to the point (1, 2).

5. Show that if an equation contains only even powers of y, the graph is symmetrical with respect to the X-axis; and if it contains only even powers of x, the graph is symmetrical with respect to the Y-axis.

6. If $y = ax^2 + bx + c$, find the coördinates of the turning point.

7. A rectangle is inscribed in a circle of radius 5. Express the area of the rectangle as a function of the length of one side. Draw the graph of the function thus found, and find the coördinates of the turning point. What is the length of the side of the rectangle of maximum area inscribed in the circle?

8. A box is to be constructed having a square base and containing 108 cubic feet. The box is to have no cover. Express the number of square feet of lumber required as a function of the length of the side of the base. Draw the graph of the function obtained and locate the turning point. What are the coördinates of the turning point? What is the size of the box requiring the least amount of lumber to construct it?

31. Graph in polar coördinates. Let r be given as a function of θ, then corresponding values of the independent variable and of the function can be regarded as polar coördinates of points. When r and θ are connected by an equation, a table of corresponding values can be computed and plotted as in rectangular coördinates. The totality of radii obtained in this way forms a graphical representation of the function, and a smooth curve drawn through the plotted points is the **graph of the function in polar coördinates.** For example, let the function be given by the equation

$$r = 2\theta.$$

The following is a table of corresponding values, θ being measured in radians:

$$\theta = 0, \quad \frac{\pi}{6}, \quad \frac{\pi}{3}, \quad \frac{\pi}{2}, \quad \frac{2\pi}{3}, \quad \frac{5\pi}{6}, \quad \pi, \quad \cdots.$$

$$r = 0, \quad 1.0472, \quad 2.0944, \quad 3.1416, \quad 4.1888, \quad 5.2360, \quad 6.2832, \quad \cdots.$$

A part of the graph is shown in Fig. 29. The curve is called a spiral of Archimedes, after its discoverer.

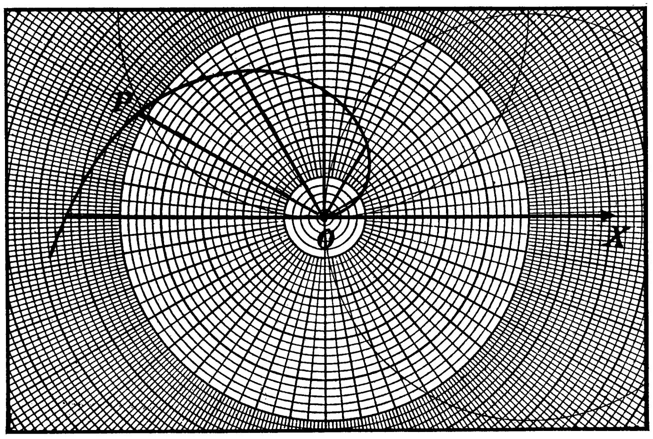


FIG. 29

EXERCISES

1. Draw the graph of the equation $r = \frac{\theta}{2}$ and compare with the graph in the preceding article.

2. Draw the graph of the function $r = \frac{2}{\theta}$. The curve is called the **reciprocal spiral.**

3. How does the graph of $r = 2\theta + 1$ differ from the graph in the preceding article?

4. If the abscissa of every point in Fig. 27, Art. 29, is diminished by $2\frac{1}{2}$ units, how will this affect the graph? How will it affect the equation? Write the new equation and draw the graph. Compare the graph with those in Arts. 27 and 28.

5. In the spiral of Archimedes, let the radius vector rotate in the negative direction. Draw the curve and compare with the graph in Art. 31.

6. Draw the graph of $r = 3\,\theta$. How does this curve differ from the spiral of Archimedes in Art. 31?

7. In Fig. 29, the curve will cross the initial line when θ is any integral multiple of π. Why? What is the distance between any two consecutive points of crossing?

8. Draw the graph of $r = a\theta$, where a is any constant number. For what values of θ does the graph cross the initial line? What is the distance between any two consecutive points of crossing?

32. Algebraic functions. If the function and the independent variable are connected by an algebraic equation, that is, an equation involving only a finite number of the fundamental operations of addition, subtraction, multiplication, division, involution, and evolution, the function is called an **algebraic function.** Thus, for example, in each of the equations $y = 2\,x^2$, $y^2 = 2\,x$, $y^2 - 5\,y + x = 0$, $x^3 - 3\,xy + y^3 = 0$, y is an algebraic function of x.

To find the value, or values, of an algebraic function for any given value of the independent variable, it is usually necessary to solve an algebraic equation. For example, if $y^2 - 5\,y + x = 0$, it is necessary to solve a quadratic equation to find the values of y for any given value of x.

33. Transcendental functions. In many cases of great practical importance, the function and the independent variable are not connected by an algebraic equation, and then the function is called a **transcendental function.** The simplest examples of transcendental functions are furnished by the trigonometric functions and logarithmic functions. Thus,

$$y = \sin x \text{ and } y = \log x$$

are transcendental functions.

To find the value of a transcendental function for a given value of the independent variable, use is made of a table. We thus have tables of logarithms and tables of trigonometric functions.

34. Graphs of transcendental functions. Corresponding values of function and independent variable can be taken directly from the table and the function exhibited graphically in rectangular coördinates or in polar coördinates, as in the preceding articles.

EXERCISES

1. Draw the graphs of the following functions. State which are algebraic functions and which are transcendental functions.

(*a*) $y = \tan x$, (*b*) $y^3 = x^2$, (*c*) $y = \cos x$,
(*d*) $y^2 = 4x^2$, (*e*) $y = \log x$, (*f*) $x^2 + y^2 = 2x$.

2. Draw the polar graphs of the following functions.

(*a*) $r = \dfrac{a}{\sin\theta}$, (*b*) $r = 2a(1 - \cos\theta)$, (*c*) $r = a(1 + \cos\theta)$.

3. Using the relations between rectangular and polar coördinates (Art. 9), change the equations in exercise 2 to rectangular coördinates and plot y as a function of x.

4. Change the equation (*f*) of exercise 1 to polar coördinates and plot r as a function of θ.

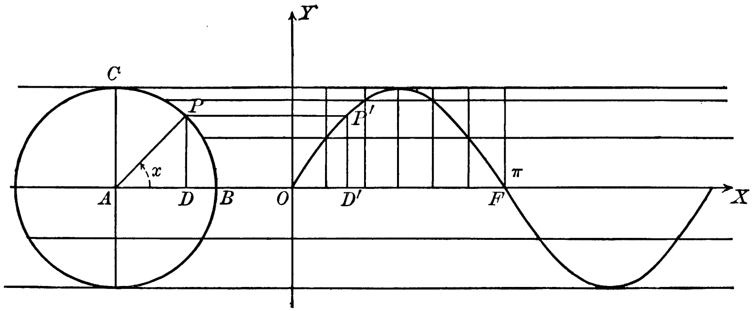


FIG. 30

35. Geometric construction of the graphs of trigonometric functions. The graphs of trigonometric functions can be constructed geometrically without the use of tables. For example, to construct the graph of $y = \sin x$, let O be the origin (Fig. 30) and F the point representing π on the scale OX. With any point A on OX as center and the unit of the scale on the Y-axis as radius, draw a circle. Let BAP be any angle x measured in degrees. The perpendicular DP is then $\sin x$. Take the distance OD' so that

$$OD' : OF :: x^\circ : 180^\circ,$$

then the point D' represents the angle x measured in radians on the scale OX. Through D' draw the perpendicular to OX and through P the parallel to OX. Let these lines meet in P', then

$D'P' = DP = \sin x$. Hence, as P describes the circle, P' describes the graph of $y = \sin x$.

For convenience, divide OF into a number of equal parts and erect perpendiculars to OX through the points of division, then divide the quadrant BPC into the same number of equal parts and draw parallels to OX through the points of division. Each perpendicular meets its corresponding parallel in a point of the graph, as indicated in the figure.

The graph of $y = \sin x$ is called the **sinusoid**, or **wave curve.**

Trigonometric functions are **periodic** functions; that is, the value of the function is repeated again and again for values of the variable which differ by a constant. Thus $\sin x$ has the same value when x is increased or decreased by any integral multiple of 2π. Many of the phenomena in nature are also periodic. For this reason, trigonometric functions are of great importance in the applications of mathematics.

EXERCISES

1. By measuring angles from the line CA, Fig. 30, instead of from the line BA, show how to construct geometrically the graph of $y = \cos x$.

2. In Fig. 30, draw the tangent to the circle at B and let it meet the radius AP produced in K. Then BK is $\tan BAP$ $(AB = 1)$; show how to construct geometrically the graph of $y = \tan x$.

3. Taking CA for the initial line, show how to construct the graph of $y = \cot x$.

4. Devise a method for constructing geometrically the graphs of $y = \sec x$ and $y = \operatorname{cosec} x$.

5. How can the graph of a trigonometric function be used to find the value of the function for any given value of the variable?

6. A point P describes a circle of radius a with the uniform velocity of k radians per second. Show that the *period*, that is, the time of one complete revolution, is $T = \frac{2\pi}{k}$.

7. Let the center of the circle in the preceding exercise be the origin of rectangular coördinates. Show that, at the end of t seconds, the coördinates of the point P are

$$x = a\cos kt = a\cos\frac{2\pi t}{T}.$$

$$y = a\sin kt = a\sin\frac{2\pi t}{T}.$$

The kind of motion described by either of these equations is called a **simple harmonic motion** (S. H. M.), a is called the **amplitude** and T the **period** of the S. H. M.

36. The exponential function. When the function and the independent variable are connected by the equation

$$y = a^x,$$

y is called an **exponential function** of x. The constant a is called the **base.** The exponential function is transcendental, since y and x are not connected by an algebraic equation.

37. Graph of the exponential function. The graph of the exponential function can be constructed as follows: From a point A on the X-axis (Fig. 31) lay off a unit AB and erect the ordinate BB_1 equal in length to the base a. Draw the line AB_1 and also the line AZ making an angle of 45° with the X-axis. Through B_1 draw the parallel to the X-axis meeting AZ in C_2, and through C_2 the perpendicular to the X-axis meeting AB_1 in C_1 and the X-axis in C. The segment CC_1 is equal in length to a^2; that is, the value of the function when x is 2. For, by similar triangles,

$$AB : BB_1 :: AC : CC_1, \text{ or}$$
$$1 : a :: a : CC_1,$$

since $AC = CC_2 = BB_1 = a$. Hence, $CC_1 = a^2$.

Similarly, drawing the parallel to the X-axis through C_1 and the perpendicular through C_3, we can prove that DD_1 is equal in length to a^3. Thus all the positive integral powers of a can be constructed geometrically. The negative integral powers can also be constructed by means of the parallels and perpendiculars. Thus, $MM_1 = a^{-1}$, $KK_1 = a^{-2}$, etc.

Let O be the origin of coördinates and OY the Y-axis. Construct parallels to the Y-axis at intervals of a unit, thus forming a series of rectangles with the parallels to the X-axis. The graph of $y = a^x$ cuts through opposite corners of these rectangles, beginning from the point (0, 1) and running each way.

The exponential function has no turning points and is therefore a monotone function (Art. 27). It is important in representing physical phenomena which are not periodic, such, for example, as the retarding effect of friction, the pressure of the atmosphere as a function of the altitude, etc. The exponential function is also important in computing interest tables, since the amount y, of \$1 for x years, at rate i compound interest, is given by the formula

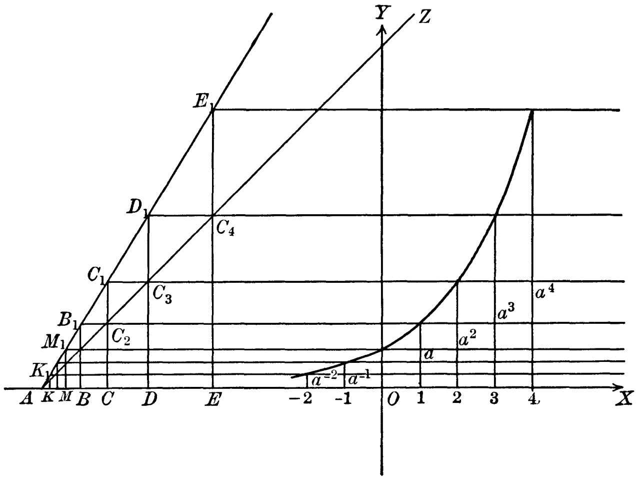


FIG. 31

$$y = (1 + i)^x.$$

Frequently the base is taken to be $e = 2.71828 \cdots$. The number e is the base of the natural, or naperian, system of logarithms.

38. Inverse functions. If two variables are connected by an equation, or otherwise, either variable may be regarded as a function of the other. For example, in the equation $y = 2x^2$, we think of y as the function and x as the independent variable, but we may regard x as the function and y as the independent variable. Either of these functions, y or x, is called the **inverse** of the other.

It is convenient to retain the notation "y means function and x means independent variable." Hence, to obtain the equation defining the inverse of a given function y, we have but to interchange x and y in the given equation and then express y in terms of x. Thus, in the above example, the inverse function is defined by the equation

$$x = 2y^2, \text{ or } y = \pm\sqrt{\frac{x}{2}}.$$

Similarly, the equations

$$y = \sin x \text{ and } x = \sin y, \text{ or } y = \text{arc} \sin x,$$

define a pair of inverse functions. Again, the equations

$$y = a^x \text{ and } x = a^y, \text{ or } y = \log_a x,$$

define a pair of inverse functions.

EXERCISES

1. Draw the graph representing the amount of \$1 at 5% compound interest as a function of the time, interest being compounded annually.

2. Show that the following pairs of equations represent inverse functions:

(*a*) $y = 3x^3$ and $y = \sqrt[3]{\frac{x}{3}}$, (*b*) $y = 5x - 6 - x^2$ and $y = \frac{5}{2} \pm \sqrt{\frac{1}{4} - x}$,

(*c*) $y = a^x$ and $y = \frac{\log_b x}{\log_b a}$, (*d*) $y = \tan 2x$ and $y = \frac{1}{2} \text{arc} \tan x$.

3. Write the inverse of each of the following functions.

(*a*) $y = \cos 3x$, (*b*) $y = \sqrt{\tan \frac{x}{2}}$,

(*c*) $y = \log_e \frac{a}{x}$, (*d*) $y = x^2 - 5x + 6$.

4. Show how to construct the graph of $y = a^{-x}$ from the graph of $y = a^x$ in Art. 37. How will changing the sign of x affect any graph?

5. Given the graphs of $y = a^x$ and $y = a^{-x}$ on the same coördinate axes, how can one construct geometrically the graph of $y = \frac{a^x + a^{-x}}{2}$?

6. With the graphs of $y = \sin x$ and $y = \cos x$ on the same coördinate axes, construct geometrically the graph of $y = \sin x + \cos x$.

39. Graph of an inverse function. Since the inverse of a given function is obtained by interchanging x and y, the graph of the inverse function can be constructed by interchanging the coördinates of every point on the graph of the given function. Thus, if P (Fig. 32) is a point on the graph of the given function, P' is a point on the graph of the inverse function when

$$OD' = DP \text{ and } D'P' = OD.$$

By this construction, P and P' are symmetrically situated with respect to the line OA bisecting the first and the third quadrants. As P describes the graph of the given function, P' describes the graph of the inverse function. Hence, having given the graph of any function, we can obtain the graph of the inverse function by plotting points symmetrically situated to the points of the given graph with respect to the line OA. Or we may consider the entire plane rotated through 180° about the line OA, carrying the given graph with it. The new position of the graph is the graph of the inverse function. For example, let MN be the graph of $y = a^x$ (Fig. 32), where a is taken to be 2. Rotating the plane about OA, the curve assumes the position RS, symmetrical to MN with respect to the line OA. Therefore RS is the graph of the inverse function $y = \log_a x$.

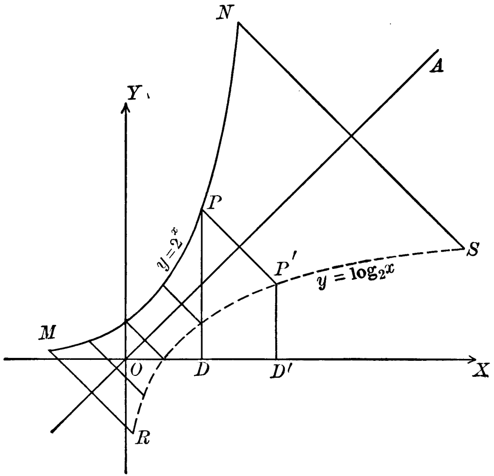


FIG. 32

EXERCISES

1. Given $y = 5x - 6 - x^2$, draw the graph of the inverse function.

2. Construct the graphs of the following functions:

(a) $y = \text{arc} \sin x$, (b) $y = \text{arc} \tan x$, (c) $y = \text{arc} \cos x$,

(d) $y = x^{\frac{2}{3}}$ and $y = x^{\frac{3}{2}}$.

3. Show that the graph of $y = \frac{a}{x}$ and the graph of the inverse function coincide throughout. What condition must be satisfied in order that the graph of any function shall coincide with the graph of its inverse?

40. Observation. A functional correspondence between two variables is often established by observation when no relation between the variables is known. Thus the temperature at a given place can be observed throughout the day and the results tabulated. The temperature can then be regarded as a function of the time, and the functional relationship can be graphically exhibited as in the preceding articles. In such cases the functional relationship is given in the form of a table of corresponding values.

41. Machines. Machines are devised to draw the graph automatically and thus avoid the necessity of making repeated observations. For example, the weather bureau has an instrument to graph the temperature as a function of the time. Coördinate paper is wound upon a clock-driven drum, and a pen is connected with a thermometer in such a way that the rise and fall of temperature is recorded upon the paper at the proper time. The record exhibits the functional relationship in the form of a graph. Corresponding values of the function and the independent variable can be read from the graph as readily as from a table.

Other records exhibit functional relationship in the form of a graph upon polar coördinate paper.

EXERCISES

1. The following table shows the length of a rubber cord in centimeters when stretched by a weight in kilograms attached to one end. Draw a curve representing approximately the graph of the length as a function of the weight.

Weight	0	.5	1.0	1.5	2.0	2.5	3.0
Length	10	10.1	10.3	10.6	10.9	11.3	11.7
Weight	3.5	4.0	4.5	5.0	5.5	6.0	
Length	12.2	12.7	13.3	13.9	14.6	15.3	

2. The number of deaths per hundred thousand lives, according to the American experience table of mortality, is as follows:

Age	Number of Deaths	Age	Number of Deaths
20	781	60	2669
25	807	65	4013
30	843	70	6199
35	895	75	9437
40	979	80	14447
45	1116	85	23555
50	1378	90	45455
55	1857	95	100000

Draw a curve representing the graph of the number of deaths as a function of the age.

3. The net annual premium for an assurance of $1000 for life, according to the American experience table of mortality, interest at 3 %, is as follows :

AGE	PREMIUM	AGE	PREMIUM
20	$14.41	40	$24.75
25	$16.11	45	$29.67
30	$18.28	50	$36.36
35	$21.08		

Draw a curve representing the graph of the premium as a function of the age.

4. The cost of a money order depends upon the amount as follows :

AMOUNT	COST	AMOUNT	COST
$0 to $2.50	3 ct.	$30 to $40	15 ct.
$2.50 to $5	5 ct.	$40 to $50	18 ct.
$5 to $10	8 ct.	$50 to $60	20 ct.
$10 to $20	10 ct.	$60 to $75	25 ct.
$20 to $30	12 ct.	$75 to $100	30 ct.

Draw a curve representing the graph of the cost as a function of the amount.

5. Figure 33 A (p. 50) represents a thermograph for April 12, 13, and 14. The ordinates are made curvilinear to allow for the pivotal motion of the drawing pen.

Determine the maxima and minima temperatures between noon of April 12 and noon of April 14. When was the temperature highest ? When lowest ?

6. Figure 33 B (p. 51) is a steam pressure gauge on polar coördinate paper. The radii are made curvilinear to allow for the pivotal motion of the drawing pen.

Determine the time of greatest pressure. The time of least pressure.

7. By means of the table of exponential functions (page V), make a careful drawing of the graph of $y = e^x$. From the graph thus made construct the graph of $y = e^{-x}$. Compare the readings from the graph with the values of e^{-x} taken from the table.

From the graph of $y = e^x$, construct the graph of $y = -e^x$.

8. Having given the graph of $y = f(x)$, show how to obtain the graphs of $y = f(-x)$, $y = -f(x)$, and $y = -f(-x)$.

9. According to Boyle's law, the volume of a gas is inversely proportional to the pressure which it sustains. If a volume of 4 cubic feet sustains a pressure of 1 atmosphere, write the equation expressing the volume as a function of the pressure. Draw the graph of this function.

10. The increase in length of a metal bar is proportional to the temperature to which the bar is subjected. If the bar is 1 foot long at 0° temperature and 1.0004 feet long at 20° temperature, write the equation expressing the length as a function of the temperature. Draw the graph of this function.

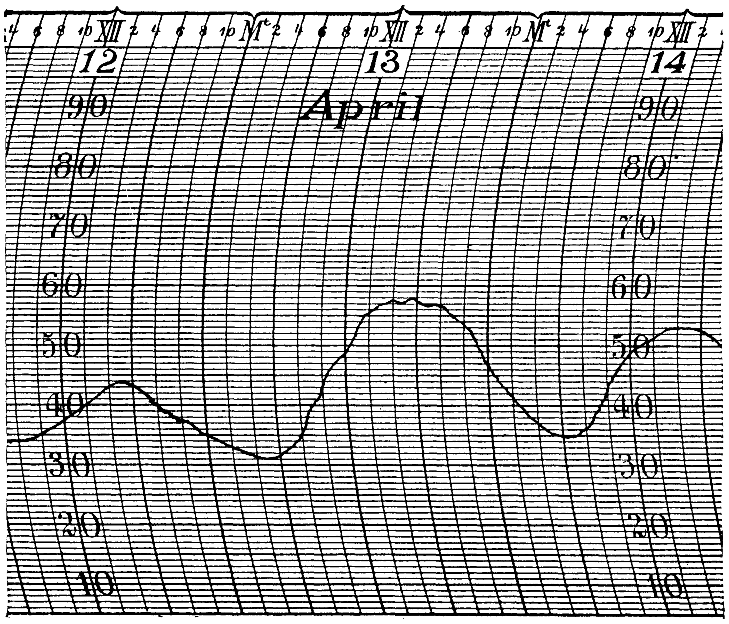


FIG. 33 A

11. The intensity of light is inversely proportional to the square of the distance from the source of the light. Write the equation expressing the intensity as a function of the distance. Draw the graph of this function. If the intensity of light at a point on the earth directly underneath the sun is taken as the unit of intensity, calculate the intensity of light on the planet Venus at a point directly underneath the sun. Take the distance from the earth to the sun as 93,000,000 miles and the distance of Venus from the sun as 67,000,000 miles.

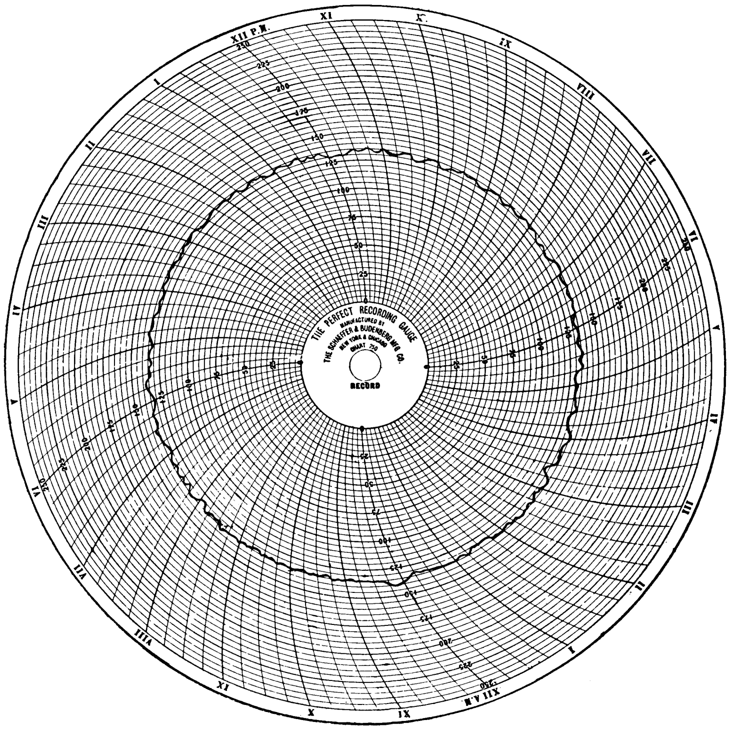


FIG. 33 B

12. The water rates of a certain city depend upon the amount consumed and are as follows:

CONSUMPTION PER DAY	RATE PER 1000 GALLONS
0 to 499 gallons	.35
500 to 999 gallons	.32
1000 to 1999 gallons	.28
2000 to 2999 gallons	.24
3000 to 3999 gallons	.20
4000 to 4999 gallons	.15
5000 to 5999 gallons	.12
6000 and over gallons	.10

Calculate the monthly (30 days) bill of a consumer and draw a curve representing the graph of the amount of the bill as a function of the number of gallons consumed per month.

13. A certain mixture of concrete contains 1.4 barrels of cement per cubic yard of concrete. If the cement costs $1.20 per barrel and the sand and crushed stone costs $2.10 per cubic yard, write an equation expressing the cost of the concrete as a function of the number of cubic yards. Draw the graph of this function.

14. Express the area of a circle as a function of the radius and draw the graph of the function.

15. Draw the graphs of $y = \sin x$ and $y = \cos x$ on the same coördinate axes. From these graphs construct the graph of the function

$$y = 2 \sin x + \cos x.$$

CHAPTER IV

LOCI AND THEIR EQUATIONS

42. Locus of a point, equation of locus. When a point $P(x, y)$ moves in the plane, the path it describes is called the **locus of the point.** The coördinates x, y are then variables (Art. 22).

If the point $P(x, y)$ moves according to a given law, this law will lead to an equation connecting x and y called the **equation of the locus.** The equation of the locus defines y as a function of x, and the locus itself is the graph of this function. As an example, suppose P moves so that it is constantly at a fixed distance from a fixed point A. We know, then, that the point describes a circle. This circle is the locus of the point P. The point P moves according to the given law

$$AP = \text{constant},$$

and we shall see that this law leads to an equation connecting the variable coördinates x and y.

43. A fundamental problem. When the law which governs the motion of a point is given, a fundamental problem presents itself; namely, *to find the equation of the locus.* For example, suppose a point moves so that it is always equidistant from the points $F \equiv (1, 2)$ and $F_1 \equiv (3, 1)$ (Fig. 34). To find the equation of the locus, let $P(x, y)$ be any point equidistant from F and F_1. Then, by the given law,

$$PF = PF_1, \tag{1}$$

for all positions of P. But

$$PF = \sqrt{(x-1)^2 + (y-2)^2} \text{ and } PF_1 = \sqrt{(x-3)^2 + (y-1)^2}.$$

Therefore we have

$$\sqrt{(x-1)^2 + (y-2)^2} = \sqrt{(x-3)^2 + (y-1)^2}, \tag{2}$$

which reduces to $4x - 2y - 5 = 0.$ (3)
This is the required equation. The locus is the perpendicular bisector of the segment FF_1.

The following property and its converse are characteristic of this locus and its equation; namely, the coördinates of every point on the locus satisfy equation (3). For, if the point is on the locus, it is equidistant from F and F_1. Therefore its coördinates satisfy (2) and consequently (3).

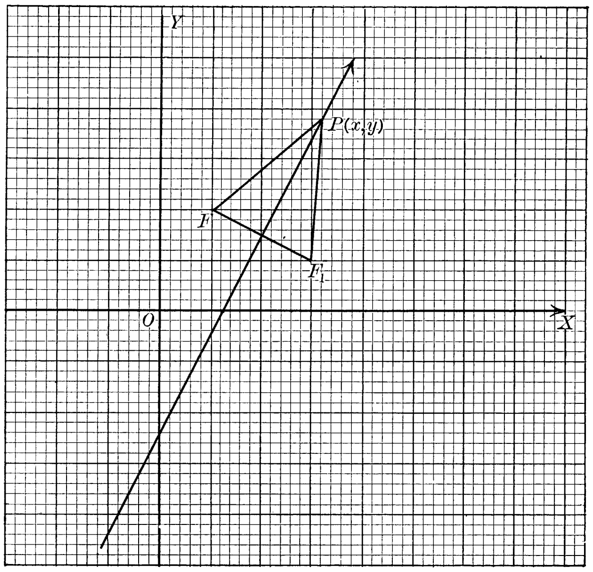


FIG. 34

Conversely, if the coördinates of any point satisfy equation (3), the point is on the locus. For then the coördinates of the point also satisfy equation (2), and the point is therefore equidistant from F and F_1; that is, the point is on the locus.

44. General definition. The property just proved for the special locus in Fig. 34 leads to the following definition: The equation of the locus of a point is an equation in the variables x and y which is satisfied by the coördinates of every point on the locus;

and conversely, every point whose coördinates satisfy the equation lies on the locus.

The locus of a point moving according to a given law is, in general, a curve, and we shall often speak of the equation of the locus as the equation of the curve.*

The problem to find the equation of the locus when the law governing the motion of the point is given will be illustrated in the succeeding articles, where the equations of a number of important curves are found and methods given for constructing the curves.

45. The circle. *A point moves so that it is always r units from a fixed point (a, b). Find the equation of the locus.*

The locus is evidently a circle whose radius is r and whose center is the point $C \equiv (a, b)$ (Fig. 35).

To find the equation of the locus, assume that $P(x, y)$ is any point r units from C. The point P is then on the locus. The statement of the law governing the motion of P is then

$$CP = r, \tag{1}$$

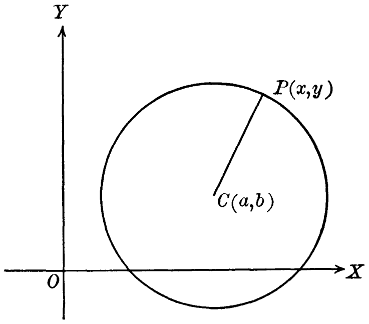


FIG. 35

for all positions of P. But

$$CP = \sqrt{(x-a)^2 + (y-b)^2}, \tag{2}$$

and therefore

$$(x-a)^2 + (y-b)^2 = r^2. \tag{3}$$

If the center of the circle is taken at the origin of coördinates, then $a = 0$ and $b = 0$, and equation (3) becomes

$$x^2 + y^2 = r^2. \tag{4}$$

Equations (3) and (4) are **standard forms** of the equation of a circle. The student should test equation (3) by the definition given in Art. 44.

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* The word "curve" will henceforth be used to denote any continuous line, straight or curved.

EXERCISES

1. Find the equations of the following circles:

(*a*) Center (0, 1) and radius 3. (*b*) Center (– 2, 0) and radius 2.

(*c*) Center (– 4, 3) and radius 3. (*d*) Center (1, 2) and radius 6.

2. Find the equation of the circle whose center is (2, 3) and which passes through the origin.

3. What is the equation of the circle which has the line joining the points (3, 2) and (– 7, 4) for a diameter?

4. Find the equation of the circle which passes through the three points (0, 1), (5, 1), and (2, – 3).

5. A point moves so as to be equidistant from the points (3, – 1) and (– 2, 3). Draw the locus and find its equation.

6. Find the equation of the perpendicular bisector of the segment joining (*a*, *b*) to (*c*, *d*).

7. A point moves so that the ratio of its distances from the points (8, 0) and (2, 0) is constantly equal to 2. Find the equation of the locus.

8. A point moves so that the sum of the squares of its distances from (3, 0) and (– 3, 0) is constantly equal to 68. Find the equation of the locus.

9. A circle circumscribes the triangle (6, 2), (7, 1), (8, – 2). Draw the figure and find the equation of the circle.

46. The equation $x^2 + y^2 + Ax + By + C = 0$. When equation (3) of the preceding article is expanded and arranged according to the powers of x and y, it takes the form

$$x^2 + y^2 + Ax + By + C = 0, \tag{1}$$

where A, B, and C are constants depending upon the radius of the circle and the coördinates of the center.

A second problem now arises: Is equation (1) the equation of a circle for all possible values of A, B, and C? To answer this question, we shall complete the squares of the terms in x and y separately, and thus put equation (1) in the form

$$\left(x + \frac{A}{2}\right)^2 + \left(y + \frac{B}{2}\right)^2 = \frac{A^2}{4} + \frac{B^2}{4} - C. \tag{2}$$

In this form, the equation states that the length of the segment joining $\left(-\frac{A}{2}, -\frac{B}{2}\right)$ to (x, y) is constantly equal to

$$\sqrt{\frac{A^2}{4} + \frac{B^2}{4} - C} \tag{3}$$

for all positions of the point (x, y). Let D stand for the expression under the radical in (3); then we can draw the following conclusions:

1. If $D > 0$, (1) is the equation of a circle whose center is the point $\left(-\frac{A}{2}, -\frac{B}{2}\right)$ and whose radius is $\sqrt{D}$.

2. If $D = 0$, (1) is satisfied by the coördinates of a single point; namely, the point $\left(-\frac{A}{2}, -\frac{B}{2}\right)$. In this case the locus is called a **null circle**.

3. If $D < 0$, there is no point in the plane whose coördinates satisfy (2) and consequently no point whose coördinates satisfy (1). In this case the locus is called an **imaginary circle**.

We shall find it convenient to say that, in any case, equation (1) is the equation of a circle, but that, in particular cases, this circle may be a null circle, or an imaginary circle.

EXERCISES

1. Find the coördinates of the center and the radius of the following circles. Construct the figure when possible.

(*a*) $x^2 + y^2 - 6x - 16 = 0$. (*b*) $x^2 + y^2 - 6x + 4y - 5 = 0$.
(*c*) $3x^2 + 3y^2 - 10x - 24y = 0$. (*d*) $(x + 1)^2 + (y - 2)^2 = 0$.
(*e*) $x^2 + y^2 = 8x$. (*f*) $7x^2 + 7y^2 - 4x - y = 3$.
(*g*) $x^2 + y^2 - 2x + 2y + 5 = 0$. (*h*) $x^2 + y^2 + 16x + 100 = 0$.

2. Find the coördinates of the center and the radius of the circle which passes through the points $(5, -3)$ and $(0, 6)$ and has its center on the line $2x - 3y - 6 = 0$.

3. A point moves so that the sum of the squares of its distances from two fixed points is constant. Prove that the locus is a circle.

4. A point moves so that the ratio of its distances from two fixed points is constant. Prove that the locus is a circle if the constant ratio is different from unity, and a straight line if the constant ratio is equal to unity.

47. The straight line. *A point moves on the straight line joining the fixed points* $P_1 \equiv (x_1, y_1)$ *and* $P_2 \equiv (x_2, y_2)$. *Find the equation of the locus.*

Choose any point $P(x, y)$ on the straight line joining P_1 to P_2 (Fig. 36). Then,

slope of segment P_1P = slope of segment P_1P_2.
But (Art. 11),

$$\text{slope of } P_1P = \frac{y - y_1}{x - x_1}, \text{ and slope of } P_1P_2 = \frac{y_2 - y_1}{x_2 - x_1}.$$

Therefore

$$\frac{\boldsymbol{y - y_1}}{\boldsymbol{x - x_1}} = \frac{\boldsymbol{y_2 - y_1}}{\boldsymbol{x_2 - x_1}}. \tag{1}$$

The expression $\frac{y_2 - y_1}{x_2 - x_1}$ is called the **slope of the line.** Representing the slope of the line by m, equation (1) becomes

$$\boldsymbol{(y - y_1) = m(x - x_1)}. \tag{2}$$

If the point $P_1 \equiv (0, b)$, that is, the point of intersection of the line with the Y-axis, equation (2) assumes the form

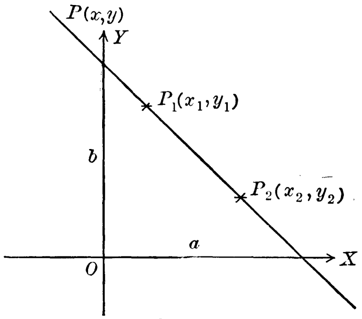

Fig. 36

$$\boldsymbol{y = mx + b.} \tag{3}$$

If $P_1 \equiv (0, b)$ and $P_2 \equiv (a, 0)$, then $m = -\frac{b}{a}$, and equation (2) reduces to

$$\boldsymbol{\frac{x}{a} + \frac{y}{b} = 1}. \tag{4}$$

Equations (1), (2), (3), and (4) are all **standard forms** of the equation of a straight line. Equation (1) is called the **two-point form**, equation (2) is the **slope-point form**, equation (3) is the **slope form**, and equation (4) is the **intercept form.**

From these equations, we conclude that *the equation of a straight line is of first degree in the variables x and y.*

Conversely, it may be shown that *any equation of the first degree in the variables x and y is the equation of a straight line.*

For, let

$$Ax + By + C = 0 \tag{5}$$

be such an equation. Solving for y, we have

$$y = -\frac{A}{B}x - \frac{C}{B}.$$

Comparison with equation (3) shows that (5) must be the equation of a straight line whose slope is $-\frac{A}{B}$ and whose intercept on the Y-axis is $-\frac{C}{B}\cdot$ This reasoning fails when B is zero. In that case, however, equation (5) reduces to $Ax + C = 0$, which is the equation of a straight line parallel to the Y-axis, since x has the constant value $-\frac{C}{A}$ for all values of y. Hence, in every case (5) is the equation of a straight line.

48. The determinant form. The equation of a straight line can be written in the form of a determinant. Thus, the equation

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0 \qquad (1)$$

is the equation of the straight line joining the points $P_1 \equiv (x_1, y_1)$ and $P_2 \equiv (x_2, y_2)$. For, equation (1) is of the first degree in x and y and therefore is the equation of some straight line, by the preceding article. Moreover, the equation states that the area of the triangle whose vertices are (x, y), (x_1, y_1), and (x_2, y_2) is zero (Art. 20). Hence, the point $P(x, y)$ is on the line joining P_1 and P_2.

EXERCISES

1. Write the equations of the lines passing through the following pairs of points:

(*a*) (0, 1) and (5, 6); (*b*) (1, − 2) and (− 3, 4); (*c*) (5, − 2) and (− 4, − 1); (*d*) (− 1, 3) and (3, − 4). Draw the figure in each case.

2. Find the intercepts which each of the lines in exercise 1 makes upon the coördinate axes. Write the equations in intercept form.

3. With the intercept on the Y-axis and the slope, write the equation of each line in exercise 1 in the slope form.

4. Write the equation of each of the lines in exercise 1 in the determinant form.

5. Find the slope and the intercepts of each of the following lines:

(*a*) $2y + 3x - 7 = x + 2.$

(*b*) $\frac{y-1}{2} = \frac{x-3}{3}.$

(*c*) $\frac{y-2}{x} = 3.$

(*d*) $\frac{x-y}{4} = \frac{2x-1}{3}.$

6. Write the equation of each of the lines in exercise 5 in the intercept form. In the slope form.

49. The ellipse. *A point moves so that the sum of its distances from two fixed points F and F_1 is constantly equal to $2a$. Construct the locus and find its equation.*

In the first place, $2a$ must be greater than the length of the segment FF_1, otherwise no locus is possible. Lay off a line A_1B_1, $2a$ units in length (Fig. 37). Take C, any point on A_1B_1, and with A_1C as radius describe a circle about F. With CB_1 as radius describe a circle about F_1. The two circles meet in the points M and N. These points are on the locus, since the sum of the radii of the two circles is $2a$. Taking the smaller circle about F_1 and the larger about F, two more points, M' and N', are found on the locus. By taking C at different places on A_1B_1, as many points of the locus can be found as may be desired.

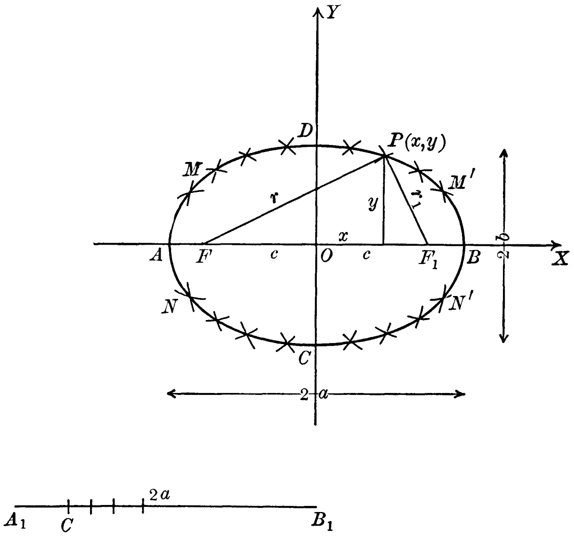

Fig. 37

Another construction of the locus is made as follows: stick pins in the paper at the points F and F_1. Tie the ends of a string together so that the loop is just equal to $2a$ plus the distance from F to F_1. Drop the loop over the pins and stretch it taut with a pencil point. Keeping the string stretched, move the pencil around; it will describe the locus.

This locus is called an **ellipse**. The fixed points F and F_1 are called the **foci** of the ellipse. The distances from any point on the ellipse to the foci are called the **focal radii** of the point.

To find the equation of the ellipse, let the line joining the foci be the X-axis, and the perpendicular bisector of the segment FF_1, the Y-axis. Let $P(x, y)$ be any point on the ellipse; then $PF = r$ and $PF_1 = r_1$ are the focal radii of P. By definition we have

$$r + r_1 = 2a \tag{1}$$

for every position of P.

Let $2c$ denote the length of the segment FF_1; then the coördinates of F and F_1 are $(-c, 0)$ and $(c, 0)$, respectively. Then

$$\begin{aligned} r^2 &= (c + x)^2 + y^2 = c^2 + 2cx + x^2 + y^2, \\ \text{and} \quad r_1^2 &= (c - x)^2 + y^2 = c^2 - 2cx + x^2 + y^2. \end{aligned} \tag{2}$$

By subtraction, we obtain

$$r^2 - r_1^2 \equiv (r - r_1)(r + r_1) = 4cx.$$

Hence, since $r + r_1 = 2a$,

$$r - r_1 = \frac{4cx}{2a} = \frac{2cx}{a}. \tag{3}$$

From (1) and (3) we get,

$$\begin{aligned} \boldsymbol{r} &= \boldsymbol{a} + \frac{\boldsymbol{cx}}{\boldsymbol{a}}, \\ \boldsymbol{r_1} &= \boldsymbol{a} - \frac{\boldsymbol{cx}}{\boldsymbol{a}}. \end{aligned} \tag{4}$$

Substituting the value of r in the first of equations (2), we obtain, after reduction,

$$\frac{x^2}{a^2} + \frac{y^2}{a^2 - c^2} = 1. \tag{5}$$

A further simplification is obtained by putting

$$a^2 - c^2 = b^2, \tag{6}$$

and then the equation assumes the final form

$$\boldsymbol{\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.} \tag{7}$$

Equation (7) is the **standard form** of the equation of an ellipse.

50. The axes and eccentricity. The segment of the line joining the foci and limited by the curve is called the **major or transverse axis** of the ellipse. That part of the perpendicular bisector of the segment joining the foci which is contained within the curve is the **minor or conjugate axis** of the ellipse. Thus, *AB* (Fig. 37) is the major axis and *CD* the minor axis. The axes intersect in the **center**, and cut the curve in the **vertices.**

When the equation of the ellipse is in the standard form, the axes of the curve coincide with the axes of coördinates (Art. 49). Hence the lengths of the axes of the ellipse can be determined from the intercepts (Art. 30) made by the curve upon the coördinate axes. From equation (7) of the preceding article we find that the intercepts on the *X*-axis are $\pm a$ and the intercepts on the *Y*-axis are $\pm b$. Therefore the length of the major axis is $2a$ and the length of the minor axis is $2b$. The segments *OB* and *OD* (Fig. 37) are called the semimajor axis and the semiminor axis, respectively.

The ratio of the distance between the foci to the length of the major axis is called the **eccentricity** of the ellipse. Since the distance between the foci is $2c$ and the length of the major axis is $2a$, the eccentricity is

$$e = \frac{c}{a}. \tag{1}$$

From equation (6) (Art. 49), $c = \sqrt{a^2 - b^2}$. Therefore

$$e = \frac{\sqrt{a^2 - b^2}}{a}. \tag{2}$$

Since a is always greater than c, the eccentricity of the ellipse is necessarily always less than unity.

Combining equations (4) Art. 49, with equation (1), we see that the lengths of the focal radii of the point $P(x, y)$ are

$$r = a + ex \text{ and } r_1 = a - ex. \tag{3}$$

EXERCISES

1. Find the equation of the ellipse for which the sum of the focal radii is 8 and the distance between the foci is 6, the origin being at the center. What is the eccentricity of this ellipse? Construct the ellipse.

2. An ellipse passes through the points $(-5, 0)$ and $(0, 3)$ and is symmetrical with respect to both axes. Find the coördinates of the foci and draw the curve.

3. Write the standard form of the equation of the ellipse having given: (*a*) the length of the transverse axis is 10 and the distance between the foci is 8; (*b*) the sum of the axes is 18 and the difference of the axes is 6; (*c*) transverse axis is 10 and the conjugate axis is $\frac{1}{2}$ the transverse axis; (*d*) transverse axis is 20 and conjugate axis is equal to the distance between the foci; (*e*) conjugate axis is 10 and distance between the foci is 10.

4. The equation of an ellipse is $\frac{x^2}{64} + \frac{y^2}{15} = 1$. Find the lengths of the focal radii of the points whose abscissa is $\frac{1}{2}$.

5. Find the lengths of the semiaxes and the eccentricity of each of the ellipses whose equations are:

$$(a)\ 3x^2 + 2y^2 = 6;\quad (b)\ \frac{x^2}{3} + \frac{y^2}{2} = 1;\quad (c)\ x^2 + 3y^2 = 2;\quad (d)\ 4y^2 + 2x^2 = 2.$$

6. The latus rectum, or parameter, of an ellipse is the double ordinate, or double abscissa, passing through a focus. Find the length of the latus rectum for each of the ellipses in exercise 5.

51. The hyperbola. *A point moves so that the difference of its distances from two fixed points F and F_1 is constantly equal to $2a$. Construct the locus and find its equation.*

Here $2a$ must be less than the length of the segment FF_1. For, if P is any point in the plane, it is shown in geometry that the difference between any two sides of the triangle PFF_1 is less than the third side.

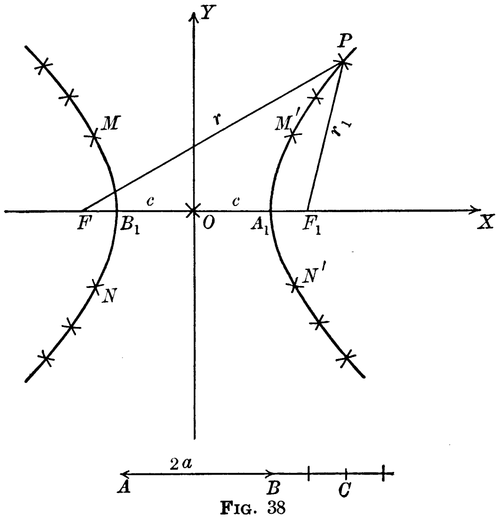

FIG. 38

Lay off a line AB (Fig. 38) $2a$ units in length and take any point C on this line produced. With BC and AC as radii and F and F_1 as centers, draw arcs of circles intersecting in M and N. These points are on the locus, since the difference of the radii of the two circles is $2a$. With the same radii, but interchanging centers, two more points, M' and N', are obtained. Taking C at different places on AB produced, as many points on the locus can be constructed as may be desired.

The locus is called an **hyperbola**, the points F and F_1 are its **foci**, and the distances from any point on the curve to the foci are called the **focal radii** of the point. The two parts of the curve are the **branches**.

To find the equation of the hyperbola, we proceed as in the case of the ellipse. Let the line joining F and F_1 be the X-axis, and the perpendicular bisector of FF_1, the Y-axis. Let F be c units to the left of the origin and F_1, c units to the right. Take $P(x, y)$, any point on the curve, and let r and r_1 be the lengths of its focal radii $(r > r_1)$. Then, by definition,

$$r - r_1 = 2a. \tag{1}$$

Equations (2) of Art. 49 hold for the hyperbola, and we obtain from them, by subtraction,

$$(r - r_1)(r + r_1) = 4cx. \tag{2}$$

Combining (1) and (2), we have

$$r + r_1 = \frac{2cx}{a}. \tag{3}$$

From (1) and (3) we get

$$\begin{aligned} \boldsymbol{r} &= \boldsymbol{\frac{cx}{a} + a}, \\ \boldsymbol{r_1} &= \boldsymbol{\frac{cx}{a} - a}. \end{aligned} \tag{4}$$

Substituting the value of r in the first of equations (2), in Art. 49, we obtain, after reduction,

$$\boldsymbol{\frac{x^2}{a^2} - \frac{y^2}{c^2 - a^2} = 1}. \tag{5}$$

A further simplification is obtained by putting

$$c^2 - a^2 = b^2 \tag{6}$$

and the equation assumes the final form

$$\boldsymbol{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1}. \tag{7}$$

Equation (7) is the **standard form** of the equation of an hyperbola

52. Axes and eccentricity. The hyperbola meets the line joining the foci in two points B_1 and A_1 (Fig. 38) which are equidistant from the mid-point O, as may be seen from the definition of the curve. The segment B_1A_1 is called the **transverse axis.** Since the intercepts on the X-axis (when the equation is in the standard form) are $\pm a$, the length of the transverse axis is $2a$.

The curve does not meet the perpendicular bisector of FF_1, since every point on this bisector is equidistant from F and F_1, but a segment extending b units above O and b units below O is called the **conjugate axis.** O is the **center** of the curve, and the transverse axis meets the curve in the **vertices,** B_1 and A_1.

The ratio of the distance between the foci to the length of the transverse axis is called the **eccentricity**. Since $FF_1 = 2c$, and $B_1A_1 = 2a$, the eccentricity is

$$\boldsymbol{e = \frac{c}{a}}. \tag{1}$$

From (6), Art. 51, we have $c = \sqrt{a^2 + b^2}$, and therefore

$$\boldsymbol{e = \frac{\sqrt{a^2 + b^2}}{a}}. \tag{2}$$

From (1) or (2) we conclude that the eccentricity of an hyperbola is always greater than unity.

Combining equations (4) of Art. 51 with equation (1), we have the lengths of the focal radii in terms of the eccentricity, namely:

$$\boldsymbol{r = ex + a} \quad \text{and} \quad \boldsymbol{r_1 = ex - a}. \tag{3}$$

EXERCISES

1. Write the standard equation of the hyperbola for which the difference between the focal radii is 6 and the distance between the foci is 8.

2. Write the standard equation of the hyperbola for which the transverse axis is 12 and the distance between the foci is 16.

3. Find the length of the focal radii of the point whose ordinate is 1 and whose abscissa is positive, the equation of the hyperbola being $\frac{x^2}{9} - \frac{y^2}{4} = 1$.

4. Find the semiaxes and eccentricity of each of the hyperbolas whose equations are:

(a) $4x^2 - 9y^2 = 36$; (b) $\frac{x^2}{4} - \frac{y^2}{9} = 1$; (c) $16x^2 - y^2 = 16$; (d) $\frac{x^2}{4} - y^2 = m$.

5. When the origin of coördinates is taken at the center of an hyperbola and the foci lie upon the Y-axis, the standard equation is $\frac{x^2}{a^2}-\frac{y^2}{b^2}=-1$. Find the lengths of the semiaxes and the eccentricity of each of the following hyperbolas:

$$(a)\ 3y^2-2x^2=12;\quad (b)\ 4x^2-16y^2=-64;\quad (c)\ y^2-mx^2=n.$$

6. The length of the double ordinate, or double abscissa, through a focus is called the **latus rectum** of the hyperbola. Find the length of the latus rectum for each of the hyperbolas in exercises 5 and 6.

53. The parabola. *A point moves so as to be equally distant from a fixed point and from a fixed straight line. Construct the locus and find its equation.*

Let F be the fixed point and AH the fixed straight line (Fig. 39). Draw AF perpendicular to AH and a series of lines parallel to AH, as PD, P_1D_1, P_2D_2, etc. With AD as radius and F as center, draw an arc cutting PD in P and Q. These points are on the locus, since $PF=AD=FQ$. Repeating the process with AD_1, AD_2, etc., as radii, a series of points on the locus is obtained.

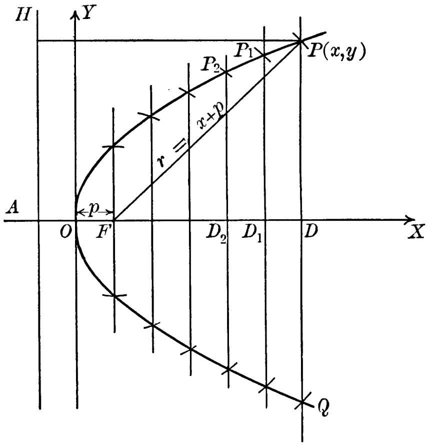

FIG. 39

The locus is called a **parabola** (cf. Art. 28). The fixed point F is called the **focus** of the parabola and the fixed line AH is called the **directrix**. The point O is the **vertex**.

To find the equation of the parabola, let AF be the X-axis and OY, the perpendicular bisector of AF, the Y-axis. Let $OF=p$, and $P(x, y)$ be any point on the curve. Then by definition,

$$PF=AD.$$

But $PF = \sqrt{(x-p)^2+y^2}$ and $AD = x + p$. Therefore,

$$\sqrt{(x-p)^2+y^2} = x + p,$$

or

$$\boldsymbol{y^2 = 4\,px.} \qquad (1)$$

Equation (1) is the **standard form** of the equation of the parabola. The number p is called the **parameter**. The distance from any point on the parabola to the focus is called the **focal radius** of the point. The length of the focal radius of any point (x, y) is

$$\boldsymbol{r = x + p.} \qquad (2)$$

EXERCISES

1. In the parabola $y^2 = 4\,x$, find the coördinates of the focus and the length of the focal radius from the point (1, 2).

2. The focus of a parabola is at the point (3, 0) and the directrix is the line $x + 1 = 0$. Find the equation.

3. The focus of a parabola is at the point (0, 2) and the directrix is the x-axis. Find the equation.

4. If the focus is 2 units from the vertex, what is the equation

(*a*) when the parabola is symmetrical with respect to the X-axis?

(*b*) when the parabola is symmetrical with respect to the Y-axis?

5. Construct each of the following parabolas:

(*a*) $y^2 = 8\,x$; (*b*) $y^2 = -4\,x$; (*c*) $x^2 = 6\,y$; (*d*) $x^2 = -10\,y$.

6. The double ordinate, or double abscissa, through the focus is called the **latus rectum** of the parabola. Find the length of the latus rectum of each parabola in exercise 5.

54. The cassinian ovals. *A point moves so that the product of its distances from two fixed points is constantly equal to* a^2. *Construct the locus and find its equation.*

Let F and F_1 be the fixed points and O the mid-point between them (Fig. 40). Draw the circle with center O and radius OF, and let FM be the tangent to this circle at F. Take FM, a units in length, and through M draw a series of secants to the circle. Let one of these secants meet the circle in the points A and A_1. Then, we have

$$MA \cdot MA_1 = \overline{FM}^2 = a^2.$$

Hence, using MA and MA_1 as radii and F and F_1 as centers, arcs of circles can be drawn intersecting in points of the locus. Thus, the points K, L, S, and T are on the locus. Repeating the process with other secants, as many points of the locus can be constructed as may be desired.

The locus is called a **cassinian oval**, after Cassini, an astronomer and engineer who lived in the latter half of the seventeenth century. The points F and F_1 are the **foci**.

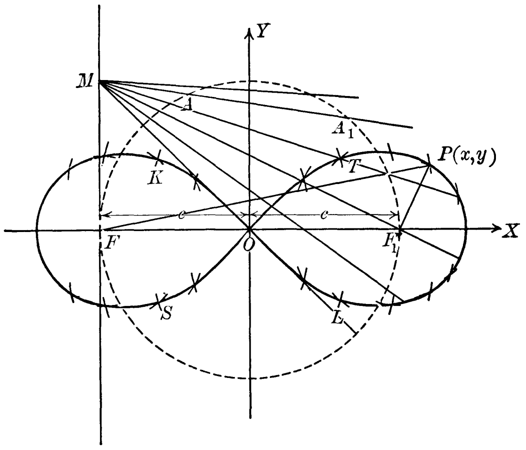

Fig. 40

To find the equation of the locus, let FF_1 be the X-axis and the perpendicular bisector of FF_1, the Y-axis. Let the distance between the foci be represented by $2c$, and let r and r_1 represent the focal radii, PF and PF_1, respectively. Then, as in Art. 49,

$$\begin{aligned} r^2 &= c^2 + 2cx + x^2 + y^2, \\ r_1^2 &= c^2 - 2cx + x^2 + y^2. \end{aligned} \tag{1}$$

Multiplying these equations, member by member, and remembering that $r \cdot r_1 = a^2$, we have

$$a^4 = (c^2 + x^2 + y^2)^2 - 4c^2x^2,$$

or

$$(x^2 + y^2)^2 - 2c^2(x^2 - y^2) = a^4 - c^4. \tag{2}$$

If $a = c$, the cassinian oval is called the **lemniscate**. This is the curve shown in Fig. 40.

EXERCISES

1. The foci of a cassinian oval are at the points $(-2, 0)$ and $(2, 0)$. Construct the curve when the product of the focal radii is 9; when the product of the focal radii is 4; when the product of the focal radii is 1.

2. Find the intercepts of a cassinian oval upon the coördinate axes, when $a > c$, when $a < c$, and when $a = c$.

3. Show that a cassinian oval is necessarily symmetrical with respect to both axes.

55. Recapitulation. The results of the preceding articles are so important that they are brought together here in compact form. The standard forms of the equations should be memorized.

Locus or Curve	Standard Forms of the Equation in Rectangular Coördinates
The straight line.	(a) Two-point form; $\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$. (b) Slope-point form; $y - y_1 = m(x - x_1)$. (c) Slope form; $y = mx + b$. (d) Intercept form; $\frac{x}{a} + \frac{y}{b} = 1$.
The circle.	(a) $(x - a)^2 + (y - b)^2 = r^2$. (b) $x^2 + y^2 = r^2$.
The ellipse.	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.
The hyperbola.	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.
The parabola.	$y^2 = 4px$.

56. Polar equation of a circle. Let $C \equiv (b, \alpha)$ be the center of a circle of radius a, and $P \equiv (r, \theta)$, any point on the circle (Fig. 41). In the triangle COP, we have $OC = b$, $OP = r$, and the angle $COP = \pm(\theta - \alpha)$, depending upon the position of P. But in either case the law of cosines applies and we have

$$r^2 + b^2 - 2br\cos(\theta - \alpha) = a^2. \tag{1}$$

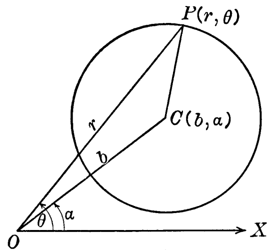

FIG. 41

This equation expresses the relation between r and θ for any point on the circle and is, therefore, the **polar equation of the circle.**

If the initial line passes through the center of the circle, $\alpha = 0$ and (1) reduces to

$$r^2 + b^2 - 2\,br \cos\theta = a^2. \qquad (2)$$

If the pole is taken on the circle, $b = a$ and (2) becomes the important form

$$r = 2\,a \cos\theta. \qquad (3)$$

This equation is also immediately deduced from Fig. 42, since XPO is a right angle.

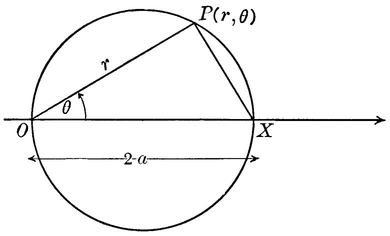

FIG. 42

If the pole is taken at the center, $b = 0$ and (1) becomes

$$r = a, \qquad (4)$$

which is the simplest form of the polar equation of a circle.

57. Polar equation of a straight line. Let AB be any straight line, O the pole, and OX the initial line (Fig. 43). Let p be the length of the perpendicular OM let fall from O upon AB, and α the angle XOM. Take P (r, θ), any point on the line AB. Then

$$r \cos(\theta - \alpha) = p \qquad (1)$$

is the **polar equation of the straight line** AB.

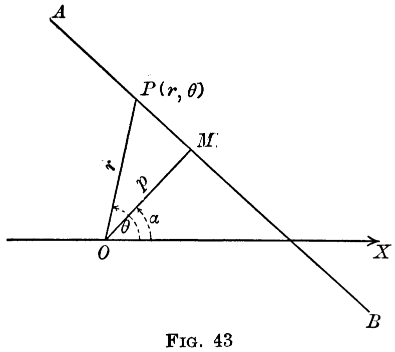

FIG. 43

EXERCISES

1. The center of a circle is at the point whose polar coördinates are $\left(3, \frac{\pi}{3}\right)$ and the radius is 4. Write the polar equation of the circle and find the length of the segment of the initial line within the circle.

2. The perpendicular from the pole upon a line is 5 units long and makes an angle of 60° with the initial line. Write the polar equation of the line. With origin at the pole and X-axis coinciding with the initial line, write the rectangular equation of the same line and find the intercepts on the axes.

3. A circle is tangent to the initial line at the pole, its radius is 4 units long, and its center lies above the initial line. What is the polar equation of the circle? What is the rectangular equation, the origin being at the pole, and the X-axis coinciding with the initial line?

4. Change the intercept form of the equation of a straight line to polar coördinates. Show that

$$a = \frac{p}{\cos \alpha} \text{ and } b = \frac{p}{\sin \alpha},$$

p and α having the same meanings as in Art. 57.

5. Discuss the polar equation of a straight line (Art. 57) for $\alpha = 0°, 90°, 180°$. Also for $p = 0$.

6. A circle passes through the origin and has its center on the line bisecting the first and third quadrants. Find the polar equation in each of the two possible positions. Also the rectangular equation.

58. Polar equation of the parabola. The polar equation of the parabola assumes the simplest form when the pole is taken at the focus and the initial line is perpendicular to the directrix (Fig. 44). Let $P(r, \theta)$ be any point on the parabola. The length of the focal radius PF is (Art. 53)

$$r = x + p. \tag{1}$$

But, from the figure,

$$x = OD = r \cos \theta + p. \tag{2}$$

Eliminating x between (1) and (2) and solving the resulting equation for r, we have

$$\mathbf{r = \frac{2p}{(1 - \cos \theta)}}. \tag{3}$$

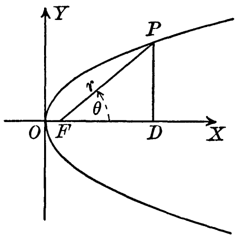

Fig. 44

59. Polar equations of the ellipse and the hyperbola. Take the pole at the left-hand focus and the initial line coincident with the transverse axis of the curve (Figs. 45 and 46). Then, for either curve, the length of the focal radius PF is given by the formula (Arts. 50 and 52)

$$r = a + ex. \tag{1}$$

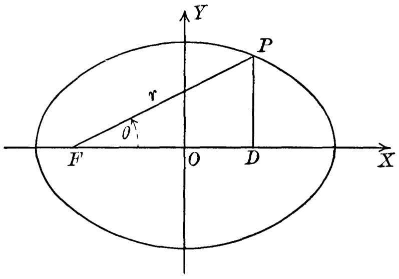

Fig. 45

But, from either figure,

$$x = OD = r\cos\theta - c. \quad (2)$$

Eliminating x between (1) and (2) and solving the resulting equation for r, we have

$$r = \frac{(a - ec)}{(1 - e\cos\theta)}. \quad (3)$$

Replacing e by its value $\frac{c}{a}$, (3) becomes

$$\boldsymbol{r = \frac{a^2 - c^2}{(a - c\cos\theta)}}. \quad (4)$$

For the ellipse, $a^2 - c^2 = b^2$ (Art. 49) and $a > c$. Hence we conclude that r is positive for all values of θ.

For the hyperbola, $a^2 - c^2 = -b^2$ (Art. 51), and $a < c$. Hence $a - c\cos\theta$ will be negative, and therefore r positive, when $\cos\theta > \frac{a}{c}$ and then the point $P(r, \theta)$ lies on the right branch of the curve. For example, when $\theta = 0$, $\cos\theta = 1$, and $r = a + c = FB$. The expression $a - c \cos\theta$ will be positive, and therefore r negative, when $\cos\theta < \frac{a}{c}$ and then the point $P(r, \theta)$ lies on the left branch of the curve. For example, when $\theta = 180°$, $\cos\theta = -1$, and $r = -(c - a) = -FA$.

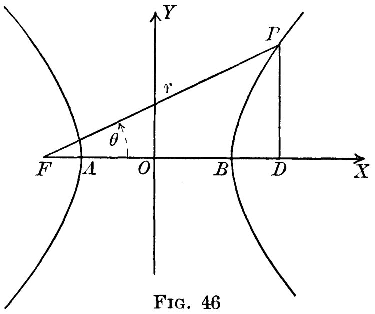

Fig. 46

EXERCISES

1. The sum of the focal radii is 8 and the distance between the foci is 6. Write the polar equation of the ellipse and sketch the curve from this equation.

2. The difference between the focal radii is 4 and the distance between the foci is 6. Write the polar equation of the hyperbola and sketch the curve from this equation.

3. Show from the polar equation that the radius vector for the ellipse is always finite in length.

4. Show from the polar equation that the radius vector for the hyperbola becomes indefinitely long for two values of θ, each less than 180°. Find these values.

5. The focus of a parabola is 6 units from the directrix. Write the polar equation and sketch the curve from this equation.

6. Show from the polar equation of the parabola that the radius vector never becomes indefinitely long except for $\theta = 2\,n\pi$, where n is any integer including zero.

7. Show that the polar equation of the lemniscate is

$$r^2 = 2\,c^2 \cos 2\,\theta,$$

the pole being at the origin and the initial line coinciding with the X-axis. Sketch the curve from this equation.

8. Change the standard forms of the equations of the ellipse, hyperbola, and parabola to polar equations, making use of equations (1), Art. 9. Why do not the equations thus found agree with the polar equations in Arts. 58 and 59?

9. Derive the polar equation of the hyperbola, assuming the right-hand focus as pole and the transverse axis as initial line.

10. Derive the polar equation of the ellipse, making the same assumptions as in the preceding exercise.

11. Compare the equation $r = \dfrac{16}{5 - 3\cos\theta}$ with equation 4, Art. 59. Does the given equation represent an ellipse or an hyperbola? What is the eccentricity and the length of the transverse axis?

12. If the semiaxes of an hyperbola are equal, the curve is called the **rectangular hyperbola**. Write the polar equation of the rectangular hyperbola.

60. Parametric equations. It is frequently useful to express the x and y coördinates of a point on a curve in terms of a third variable called the **parameter**. For example, the x and y coördinates of a point on the circle

$$x^2 + y^2 = a^2 \tag{1}$$

can be expressed as follows:

$$x = a\cos t, \quad y = a\sin t, \tag{2}$$

since these values of x and y satisfy (1), whatever value is given to the parameter t. Equations (2) are **parametric equations** of the circle whose equation in rectangular coördinates is (1).

Similarly, a pair of parametric equations of the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \tag{3}$$

are

$$x=a\cos t,\quad y=b\sin t, \tag{4}$$

since these values of x and y satisfy (3) for all values of t.

A variety of pairs of parametric equations can be found expressing the same relation between x and y. For example,

$$x=\frac{a(1-t^2)}{1+t^2} \text{ and } y=\frac{2at}{1+t^2}$$

are parametric equations of the circle (1), since these values of x and y will satisfy (1) for all values of t.

EXERCISES

1. Show that $x = a\sec t$ and $y = b\tan t$ are parametric equations of an hyperbola.

2. Show that $x=\frac{b}{m}+t$ and $y=b+mt$ satisfy the slope form of the equation of a straight line for all values of t.

3. Eliminate t from the equations $x=t^2$ and $y=2t$ and thus show that these equations are parametric equations of a parabola.

4. Write a pair of parametric equations for the standard form $y^2=4px$.

5. Show that $x=at$ and $y=b(1-t)$ are parametric equations of a straight line.

6. Show that the equations

$$x=\frac{c\sqrt{2}(1+t^2)t}{1+t^4},$$

$$y=\frac{c\sqrt{2}(1-t^2)t}{1+t^4}$$

are parametric equations of the lemniscate (Art. 54).

7. Write the parametric equations of the rectangular hyperbola

$$x^2-y^2=a^2.$$

61. Geometrical construction of the ellipse and the hyperbola. The ellipse and the hyperbola can be constructed easily by means of parametric equations (Figs. 47 and 48).

Draw the concentric circles whose radii are the semiaxes a

and b. These circles are called the **major and minor auxiliary** circles, respectively.

For the ellipse, with any value of t, construct $OD = a \cos t$ and $EP'' = b \sin t$. These are the coördinates of the point P on the ellipse.

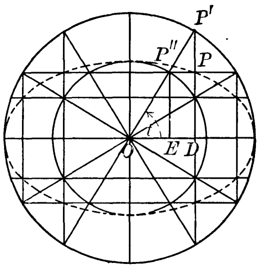

Fig. 47

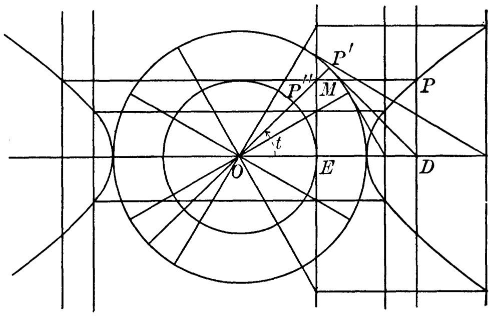

Fig. 48

Similarly, for the hyperbola, $OD = a \sec t$ and $EM = b \tan t$ are the coördinates of the point P on the curve.

The points P, P', and P'' are called **corresponding points**. As the radius OP revolves about O, the points P' and P'' move on their respective auxiliary circles, and P describes the ellipse in Fig. 47 and the hyperbola in Fig. 48.

EXERCISES

1. Write the parametric equations and construct the ellipse whose semiaxes are 3 and 4.

2. Write the parametric equations and construct the hyperbola whose semiaxes are 3 and 4.

3. Construct the following loci by assigning arbitrary values to the parameter t and tabulating the corresponding values of x and y:

$$(a)\ x = t - 1,\ y = 4 - t^2; \quad (b)\ x = \frac{t^3}{2},\ y = \frac{t}{4}; \quad (c)\ x = 3t,\ y = 3t^2 - t^3.$$

4. With θ as the parameter, construct the locus

$$x = 6 \cos \theta + 3 \cos 2\theta,$$
$$y = 6 \sin \theta - 3 \sin 2\theta.$$

This locus is called the three-cusped hypocycloid.

62. Recapitulation.

LOCUS OR CURVE	POLAR EQUATION	PARAMETRIC EQUATIONS
The circle.	$r = 2a\cos\theta.$	$x = a\cos t,\ y = a\sin t.$
The ellipse.	$\dfrac{a^2 - c^2}{a - c\cos\theta} = r,\ a > c.$	$x = a\cos t,\ y = b\sin t.$
The hyperbola.	$\dfrac{a^2 - c^2}{a - c\cos\theta} = r,\ a < c.$	$x = a\sec t,\ y = b\tan t.$
The parabola.	$\dfrac{2p}{1 - \cos\theta} = r.$	$x = \dfrac{t^2}{4p},\ y = t.$

EXERCISES

1. Find the lengths of the axes, the distance between the foci, and the eccentricity of each of the following curves.

(*a*) $9y^2 + 4x^2 = 36.$
(*b*) $7x^2 + 11y^2 = 15.$
(*c*) $100y^2 - 25x^2 = -2500.$
(*d*) $17x^2 - 25y^2 = -116.$
(*e*) $64y^2 + 25x^2 = 1600.$
(*f*) $64y^2 - 25x^2 = -1600.$

2. Show that the points $(-4, -2)$, $(2, 1)$, $(-6, 3)$, $(0, 0)$, and $(2, -1)$ lie upon two straight lines. What are the equations of these lines?

3. The semiaxes of an ellipse are 6 and 4. Find the length of the latus rectum.

4. Write the polar equation of the hyperbola, if the transverse axis is 6 and the distance between the foci is 10. For what values of θ is r infinite?

5. If the perpendicular to the major axis of an ellipse at the point D meets the major auxiliary circle in P and the ellipse in P', prove that

$$DP : DP' :: a : b,$$

where a and b are the semiaxes.

6. In geometry it is shown that the areas of rectangles having the same width are to each other as their lengths. Combining this proposition with that in the preceding exercise, show that the area of the major auxiliary circle is to the area of the ellipse as a is to b, and hence the area of the ellipse is πab.

7. If the major auxiliary circle is rotated around the major axis of the ellipse until its plane makes an angle whose cosine is $\frac{b}{a}$ with the plane of the ellipse, and if perpendiculars be dropped from every point of the circle upon the plane of the ellipse, show that the feet of these perpendiculars lie upon the ellipse.

CHAPTER V

EQUATIONS AND THEIR LOCI

63. Locus of an equation. The curve which passes through all the points whose coördinates satisfy a given equation, and through no other points, is called the **locus** of the given equation.

64. A second fundamental problem. In the last chapter we have found the equations of a number of important loci from given laws. There now arises a second fundamental problem of analytic geometry; namely, *given an equation connecting the variables x and y, to construct the locus of the equation and to discover the important properties of the locus.*

In simple cases the general form of the locus can be determined by plotting (Art. 27). But this method alone often fails to reveal the important properties of the locus, and, at best, leaves wholly undetermined the form of the locus between consecutive points plotted. A discussion of the given equation, however, will reveal certain properties of the locus which, together with a few plotted points, will determine frequently the form and nature of the locus.

65. Discussion of an equation. The method to be followed must depend upon the particular equation under discussion, but the following outline will serve to indicate what to look for in any given case.

(*a*) *Symmetry* (cf. Art. 29).

(1) If the given equation contains only even powers of y, the locus is symmetrical with respect to the X-axis. For then, if $P(a, b)$ is any point on the locus, $Q(a, -b)$ is also on the locus.

(2) If the given equation contains only even powers of x, the locus is symmetrical with respect to the Y-axis. For then, if $P(a, b)$ is any point on the locus, $Q(-a, b)$ is also on the locus.

(3) If the given equation contains only even powers of x and of

y, the locus is symmetrical with respect to the origin. For then, if $P(a, b)$ is any point on the locus, $Q(-a, -b)$ is also on the locus.

(4) If the given equation is unaltered when x and y are interchanged, the locus is symmetrical with respect to the line bisecting the first and third quadrants. For then, if $P(a, b)$ is any point on the locus, $Q(b, a)$ is also on the locus (cf. Art. 39).

(b) *Intercepts* (cf. Art. 30). The determination of the intercepts furnishes a good point from which to begin the construction of the locus.

(c) *Limits of the locus.* It frequently happens that to certain values of either variable there correspond no real values of the other. There is no corresponding real point in such a case. Hence, *the locus is confined to those parts of the plane such that to each value of either variable there corresponds a real value of the other.* For example, the locus of the equation $y^2 = 2x$ (Art. 28) is confined to the part of the plane to the right of the Y-axis.

Whenever to any value of either variable there corresponds no real value of the other, the locus is said to be limited. The equation of a limited locus, if it is algebraic, must establish at least one of the variables as a multiple-valued function of the other. For in no other way can imaginary values appear. The converse does not hold, for an equation which establishes one variable as a multiple-valued function of the other does not necessarily have a limited locus. Thus, the equation $y^2 = x^2$ establishes y as a multiple-valued function of x, but for no value of either variable is the other ever imaginary.

(d) *Change of one variable due to a given variation of the other* (Art. 27). It is important to determine from the equation how increasing or decreasing one variable will affect the other. For example, if x is allowed to increase in value, will y increase or decrease in value? In other words, to determine whether y is a monotone function or not; and, if not, for what values of x it is increasing, for what values it is decreasing, and, if possible, for what values of x it has turning points.

(e) *Behavior of the locus at great distances from the origin.*

It is also important to determine from the equation how increasing either variable indefinitely will affect the other.

The discussion of an equation according to the foregoing outline will be illustrated in the following examples.

66. Example I. Discuss the equation $x^2 + 4y^2 = 4$ and construct the locus.

(*a*) Assume any point $P(a, b)$ whose coördinates satisfy the given equation (Fig. 49). Then the coördinates of the points $Q(a, -b)$, $S(-a, b)$, and $R(-a, -b)$ also satisfy the equation. Hence the locus is symmetrical with respect to both axes, and also with respect to the origin.

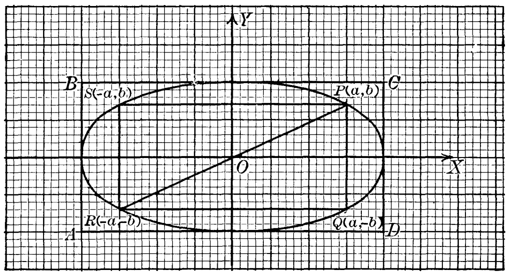

Fig. 49

(*b*) The x-intercepts are ± 2 and the y-intercepts are ± 1.

(*c*) Solving the equation for y, we have

$$y = \frac{\pm\sqrt{4 - x^2}}{2} \tag{1}$$

from which it is seen that x is limited to the range of values extending from -2 to $+2$ in order that y may have real values. The range of values for x is indicated by writing

$$-2 \leqq x \leqq +2.$$

The locus is thus limited to lie between the lines $x = -2$ and $x = +2$, or the lines AB and CD in the figure.

Again, solving the equation for x, we obtain

$$x = \pm 2\sqrt{1 - y^2}. \tag{2}$$

Hence y is limited to the range

$$-1 \leqq y \leqq 1$$

in order that x may have real values. The locus, therefore, lies between the lines $y = +1$ and $y = -1$, or the lines BC and AD in the figure. The locus, therefore, lies wholly within the rectangle $ABCD$.

(*d*) From equation (1) it follows that as x increases or decreases from zero, the positive value of y decreases, and the negative value increases.

Hence we conclude that $+2$ is a maximum value of y, and -2 is a minimum value.

Similar conclusions with respect to the values of x can be drawn from equation (2).

The foregoing discussion reveals the general form and properties of the locus. The curve is an ellipse.

67. Example II. Discuss the equation $x^2 - 4y^2 = 4$ and find the form and general properties of the locus.

(*a*) The locus is symmetrical with respect to each axis and with respect to the origin as in Example I.

(*b*) The intercepts on the X-axis are ± 2; the locus does not meet the Y-axis.

(*c*) Solving the equation for y, we have

$$y = \frac{\pm\sqrt{x^2 - 4}}{2}. \tag{1}$$

Here y is imaginary for all values of x within the range from -2 to $+2$. Hence the range for x is

$$-2 \geqq x \geqq 2$$

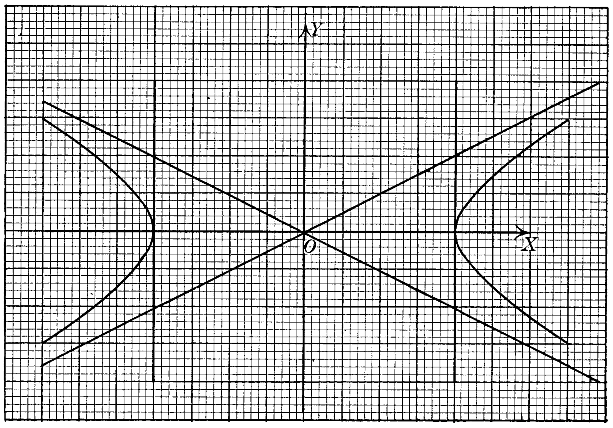

Fig. 50

in order that y may have real values. The locus, therefore, lies outside the strip bounded by the lines $x = -2$ and $x = +2$ (Fig. 50).

Solving the equation for x, we have

$$x = \pm 2\sqrt{1 + y^2}.$$

Hence x is real for all values of y. The locus is therefore unlimited in the y-direction.

(d) From equation (1), as x increases from 2, the positive value of y increases continually and without limit, but as x increases from a large negative value to -2, the positive value of y continually decreases to zero. Combining these facts with the symmetry in (a), we conclude that the locus spreads out as it recedes from the origin in either direction.

(e) As x increases indefinitely, the values of y approach nearer and nearer to $\pm \frac{x}{2}$. For the radical $\sqrt{x^2 - 4}$ is clearly always less than x in value, but for very great values of x, the difference

$$x - \sqrt{x^2 - 4} = \frac{4}{x + \sqrt{x^2 - 4}},$$

is very small and can be made as small as we please by choosing x sufficiently great. Therefore, at great distances from the origin, the locus lies close to the straight lines

$$y = \pm \frac{x}{2}.$$

These lines are called **asymptotes**. The X-axis bisects one of the angles between the asymptotes and the locus lies within this angle, one branch on each side of the origin. The curve is an hyperbola.

EXERCISES

1. Discuss the following equations and draw the corresponding loci:

(a) $4x^2 + 9y^2 = 36$. (b) $4x^2 - 9y^2 = 36$. (c) $y^2 = 16x$. (d) $x^2 = 9y$.
(e) $x^2 - y^2 = 4$. (f) $x^2 + y^2 = 4$. (g) $y^2 = 4x^2$.

2. Find the lengths of the axes, the distance between the foci and the eccentricity of the ellipse in exercise 1.

3. Find the lengths of the axes, the distance between the foci, and the eccentricity of the two hyperbolas in exercise 1.

4. Find the equations of the asymptotes for each of the hyperbolas in exercise 1.

5. Find the coördinates of the focus for each of the parabolas in exercise 1.

68. Example III. Discuss the equation $xy - x - y = 0$ and find the form and properties of the locus.

(a) The locus is obviously not symmetrical with respect to either axis nor with respect to the origin. It is, however, symmetrical with respect to a line bisecting the first and third quadrants, since if $P(a, b)$ is any point whose coördinates satisfy the equation, then the coördinates of the point $Q(b, a)$

also satisfy the equation. The points P and Q are symmetrically situated with respect to the line OA (Fig. 51).

(*b*) The locus crosses the coördinate axes only at the origin. The intercept on each axis is therefore zero.

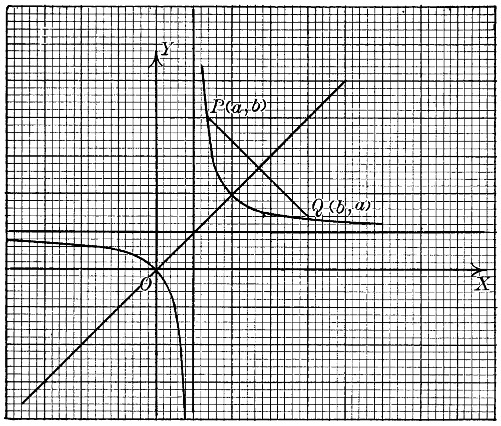

Fig. 51

(*c*) The locus is not limited in either direction, since each variable is real for all values of the other.

(*d*) Solving the equation for y, we have

$$y = \frac{x}{x-1}. \tag{1}$$

Hence, as x increases from a large negative value, y continually decreases from a value less than 1, through zero, becoming $-\infty$ for x equal to 1. As x passes the value 1, y changes suddenly to a very great positive value and then continually decreases, approaching nearer and nearer to 1. The function y is therefore monotone. It has a *discontinuity* at $x = 1$.

(*e*) From equation (1) we see that y approaches nearer and nearer to 1 as x increases or decreases indefinitely. For very great values of x, therefore, the locus lies close to the line $y = 1$. This line is an asymptote to the curve.

Similarly, solving the equation for x, the line $x = 1$ is found to be an asymptote; that is, for very great values of y the locus lies close to this line.

The above discussion enables us to form a fairly accurate idea of the locus before any plotting has been done.

69. Example IV. Discuss the equation $y = \frac{x}{1+x^2}$ and find the form and properties of the locus.

(*a*) The locus is not symmetrical with respect to either axis, but it is symmetrical with respect to the origin, since if $P(a, b)$ is any point on the locus, so also is the point $Q(-a, -b)$ on the locus (Fig. 52).

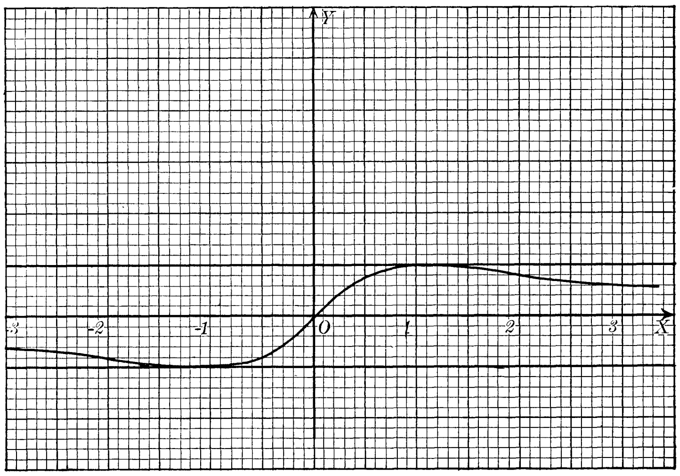

Fig. 52

(*b*) The locus crosses the axes only at the origin.

(*c*) The locus is unlimited in the x-direction, since y is real for all values of x. But if we solve the equation for x, we obtain

$$x = \frac{1 \pm \sqrt{1 - 4y^2}}{2y}$$

and therefore y is limited to the range

$$-\tfrac{1}{2} \leqq y \leqq \tfrac{1}{2}$$

in order that x may have real values. Consequently the locus lies within the strip bounded by the lines $y = -\frac{1}{2}$ and $y = +\frac{1}{2}$.

(*d*) and (*e*). As x increases from zero to 1, y increases from zero to $\frac{1}{2}$; and as x increases indefinitely from 1, y decreases slowly from $\frac{1}{2}$ towards zero. Hence the function y has a turning point at $x = 1$ and its value there is $\frac{1}{2}$. Also for very great positive values of x, the locus lies close to the X-axis. Since the locus is symmetrical with respect to the origin, its form to the left of the origin is known as soon as its form to the right has been determined. We conclude, therefore, that the function has a turning point at $x = -1$ and that the X-axis is an asymptote to the curve.

70. Example V. Discuss the equation $y^2 = \dfrac{x^2(b - x)}{3x + b}$ and find the form and properties of the locus.

(*a*) The locus is clearly symmetrical with respect to the X-axis; it is not symmetrical with respect to the Y-axis, since the equation contains odd powers of x.

(*b*) For the purpose of discussion we will suppose that b is a positive number (Fig. 53). The locus crosses both axes at the origin and also crosses the X-axis at $x = b$.

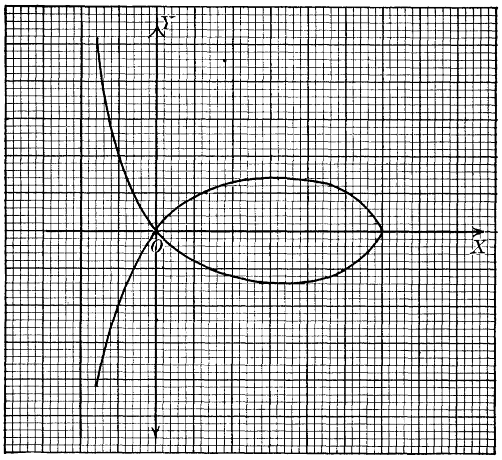

FIG. 53

(*c*) From the equation we see that x is limited to the range

$$-\frac{b}{3} \leqq x \leqq b$$

in order that y may have real values. The locus therefore lies between the lines $x = -\frac{b}{3}$ and $x = b$.

(*d*) As x increases from zero to b, the absolute value of y at first increases and then decreases to zero. This shows that the locus has a loop at the right of the origin. As x decreases from zero to $-\frac{b}{3}$, the absolute value of y increases very rapidly from zero, becoming infinite at $x = -\frac{b}{3}$. The line $x = -\frac{b}{3}$ is an asymptote to the curve. The locus is called the **folium of Descartes**.

EXERCISES

1. Discuss the following equations and plot the corresponding loci. Find the asymptotes when these exist.

(*a*) $xy - x + y = 0$. (*b*) $y^2 - 4x^2 = 4$. (*c*) $x^{\frac{1}{2}} + y^{\frac{1}{2}} = a^{\frac{1}{2}}$.

(*d*) $x^2y^2 = (y + 2)^2(9 - y^2)$. (*e*) $y = \frac{x}{1 - x^2}$. (*f*) $y = \frac{1}{(x - 2)^2}$.

2. Find the lengths of the semiaxes, the coördinates of the foci, and the eccentricity of the hyperbola whose equation is (*b*) of the previous exercise.

NOTE. The locus (*d*) in exercise 1 is called the **conchoid of Nicomedes.**

71. Example VI. Discuss the equation of the **catenary**; namely, $y = \frac{a}{2}(e^{\frac{x}{a}} + e^{-\frac{x}{a}})$, and plot the locus.

The equations discussed in the foregoing examples are algebraic equations and the corresponding loci are called *algebraic curves.* The locus of a transcendental equation is called a *transcendental curve.* Thus the catenary is a transcendental curve.

(*a*) The locus is symmetrical with respect to the Y-axis, since changing the sign of x does not alter the equation. The locus is not symmetrical with respect to the X-axis, since for every value of x, y is positive. The curve therefore lies entirely above the X-axis.

(*b*) The curve meets the Y-axis a units above the origin. It does not meet the X-axis.

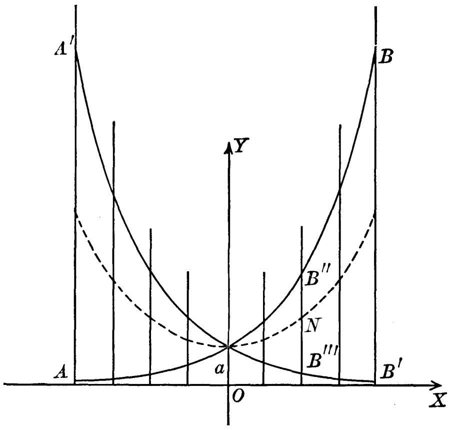

FIG. 54

(*c*) The locus is unlimited in the x-direction, since y is real for every value of x.

(*d*) and (*e*) As x increases from zero, y also increases, since neither $e^{\frac{x}{a}}$ nor $e^{-\frac{x}{a}}$ can ever become negative. We conclude, therefore, that the function y has a turning point at the origin and that its value there is a minimum.

To plot the locus, we construct the auxiliary curves

$$y_1 = e^{\frac{x}{a}} \text{ and } y_2 = e^{-\frac{x}{a}}$$

as in Art. 37, taking a for the unit of measure. These curves are respectively AB and $A'B'$ (Fig. 54). The required locus,

$$y = \frac{y_1 + y_2}{2},$$

bisects the segment of each ordinate contained between the auxiliary curves.

The catenary is the curve formed by a flexible chain hung between two supports. It is of great importance in problems connected with the construction of suspension bridges.

72. Simple harmonic curves, compound harmonic curves. The loci of the equations

$$y = a\sin\frac{2\pi x}{T} \text{ and } y = a\cos\frac{2\pi x}{T}$$

are called **simple harmonic curves.** They are constructed as in Art. 35. Simple harmonic curves represent simple vibratory or wave motion like that of a swinging pendulum carried forward with a uniform velocity in a straight line perpendicular to the plane in which the pendulum is swinging. The number a is called the **amplitude** of the vibration, and T, the **period** (cf. exercises, Art. 35).

The loci of equations of the form

$$y = a\sin\frac{2\pi x}{T_1} \pm b\sin\frac{2\pi x}{T_2} \text{ or } y = a\sin\frac{2\pi x}{T_1} \pm b\cos\frac{2\pi x}{T_2}$$

are called **compound harmonic curves.** To construct a compound harmonic curve, we plot each of the simple harmonic curves of which it is composed on the same coördinate axes and take the algebraic sum of the ordinates for any particular value of x as the ordinate of the required curve for that value of x.

In general, to construct the locus of an equation of the form

$$y = f_1(x) \pm f_2(x),$$

plot each of the auxiliary curves

$$y_1 = f_1(x) \text{ and } y_2 = f_2(x)$$

on the same coördinate axes and take the algebraic sum of the ordinates for any particular value of x as the ordinate of the required locus for that value of x.

Compound harmonic curves occur in the theories of sound, light, and electricity. Several simple harmonic curves may be combined to form a compound harmonic curve.

EXERCISES

1. If a pendulum makes 4 complete vibrations per second, show that its period is $T = \frac{1}{4}$. If the amplitude of the vibration is 2, show that the motion of a point on the pendulum is given by the equation $y = 2 \sin 8 \pi x$, where x represents time measured in seconds. Construct the locus of the equation.

2. Construct the loci of the following equations:

(a) $y = e^x + \sin x$.	(b) $y = x + \sin x$.	(c) $y = 2x - \cos x$.
(d) $y = \sin x + \sin 2x$.	(e) $y = x^2 + 2^x$.	(f) $y = x - \sin 2x$.

3. The piston of an engine is connected to the drive wheel by a connecting rod. If the crank pin describes a circle whose radius is 2 feet and makes 200 revolutions per second, what is the amplitude and the period of the harmonic motion described by the piston? Write the equation expressing this motion.

73. Damped vibrations. The loci of equations of the form

$$y = ae^{-c^2x} \sin kx \text{ or } y = ae^{-c^2x} \cos kx$$

represent damped vibrations such as a pendulum vibrating in a resisting medium. To illustrate the method of plotting the loci of such equations, we will construct the locus of the equation

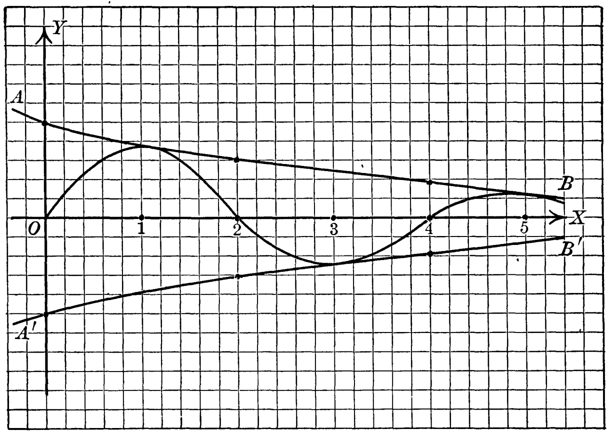

FIG. 55

$$y = e^{-\frac{1}{4}x} \sin \frac{\pi x}{2}.$$

Since the absolute value of the sine can never exceed unity, we see that the absolute value of y can never exceed the value of $e^{-\frac{1}{4}x}$. Again, when x is any odd integer, $\sin\frac{\pi x}{2}$ is either $+1$ or -1, and when x is an even integer, $\sin\frac{\pi x}{2}$ is zero. Hence we conclude that the required locus lies between the two curves

$$y = e^{-\frac{1}{4}x} \text{ and } y = -e^{-\frac{1}{4}x} \tag{1}$$

and crosses the X-axis whenever x is an even integer.

The two curves in (1) can be constructed as in Art. 37, and are called **boundary curves.** The locus is shown in Fig. 55, where AB and $A'B'$ are the boundary curves.

In general, the loci of equations of the form

$$y = f(x) \sin kx \text{ or } y = f(x) \cos kx$$

can be constructed by first plotting the loci of the boundary curves

$$y = f(x) \text{ and } y = -f(x).$$

EXERCISES

1. Construct the following loci:

(*a*) $y = x \sin x$.
(*b*) $y = x \cos x$.
(*c*) $y = \frac{x}{3} \sin \frac{\pi x}{3}$.
(*d*) $y = \frac{1}{x} \sin x$.
(*e*) $y = x^2 \sin x$.
(*f*) $y = e^{-x} \sin x$.
(*g*) $y = e^{x} \sin x$.
(*h*) $y = \left(3 + \frac{x^2}{16}\right) \sin \frac{\pi x}{2}$.

2. Discuss the equation $y^2 = x \sin^2 x$ and construct the locus.

74. Polar equations. When the given equation is in polar coördinates, the main facts about the locus can also be determined by a discussion of the equation. The points to be determined by the discussion are the following:

(*a*) *Symmetry with respect to the pole.*

(1) The locus is symmetrical with respect to the pole if, for any given value of θ, the equation is satisfied by both $+r$ and $-r$. This will happen when the equation contains only even powers of r.

(2) The locus is symmetrical with respect to the pole if, whenever the equation is satisfied by a point (r, θ), it is also satisfied by $(r, \theta + 180°)$. For then the locus cuts each radius vector at points equidistant from the pole.

(*b*) *Points where the locus crosses the initial line.* These are found by putting $\theta = 0$, or 180°, and solving the resulting equation for r. If the equation obtained by putting $r = 0$ in the given equation is satisfied by some value, or values, of θ, then the locus passes through the pole.

(*c*) *Limits of the locus.* These are determined by finding the ranges of values of each variable for which the other has real values.

(*d*) *Change of one variable due to a given variation of the other.* It is important to determine from the equation how increasing or decreasing either variable will affect the other.

75. Example IX. Discuss the equation $r = \cos 2\theta$.

(*a*) Since $r = \cos 2\theta = \cos 2(\theta + 180°)$, the locus is symmetrical with respect to the pole.

(*b*) When θ equals zero or 180°, $r = 1$. Hence the locus crosses the initial line on opposite sides of the pole and at a unit's distance from the pole. Again, $r = 0$ for $\theta = 45°, 135°, 225°$, or 315°. Therefore the locus passes through the pole four times during one complete revolution of the radius vector.

(*c*) The radius vector is real for every value of θ, but since $\cos 2\theta$ can never be greater than unity, the locus is entirely contained within the circle whose radius is 1.

(*d*) As θ varies from $-45°$ to $+45°$, r is positive and the point (r, θ) describes the loop to the right of the pole. Between 45° and 135°, $\cos 2\theta$, and therefore r, is negative and the point (r, θ) describes the loop below the pole. Between 135° and 225° r is again positive and the point (r, θ) describes the loop to the left of the pole.

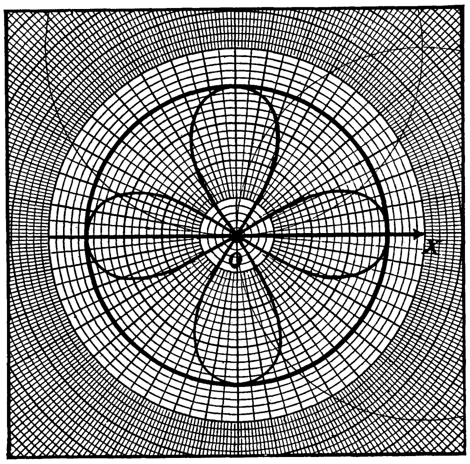

FIG. 56

Finally, between 225° and 315°, r is negative and the point (r, θ) describes the loop above the pole.

The locus is one of a family of curves known as "rose curves" from the form (Fig. 56).

76. Example X. Discuss the equation $r^2 = a^2 \cos 2\theta$.

(a) The locus is symmetrical with respect to the pole, since the equation contains only the second power of r.

(b) The locus crosses the initial line at the points for which $r = \pm a$ and also at the pole.

(c) The radius vector can be real only when θ has a value between $-45°$ and $+45°$ or between 135° and 225°. For all other positions of the radius vector, $\cos 2\theta$ is negative and consequently r is imaginary.

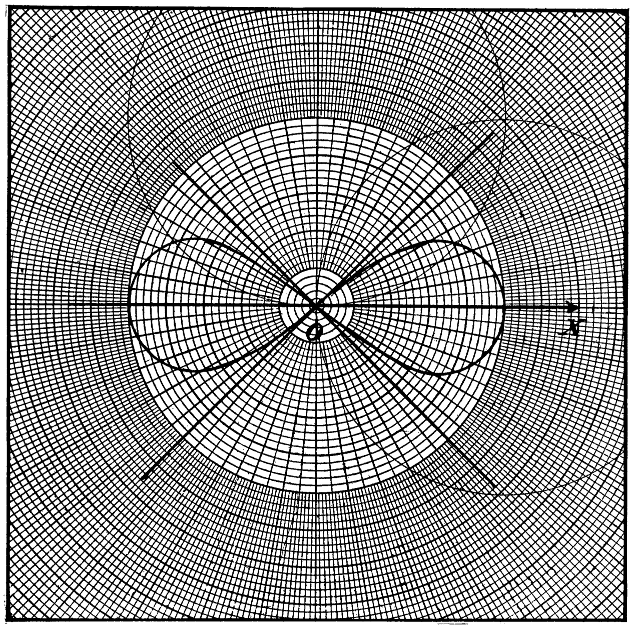

Fig. 57

(d) As θ increases from 0° to 45°, the absolute value of r decreases from a to 0. Again, as θ increases from $-45°$ to 0° the absolute value of r increases from 0 to a. The locus consists, therefore, of two loops as shown in Fig. 57.

EXERCISES

1. Discuss the following equations and construct the corresponding loci:

(*a*) $r = a \cos 3\theta$. (*b*) $r = a \sin 3\theta$. (*c*) $r = 1 + \cos\theta$.

(*d*) $r = \dfrac{1}{1 + \cos\theta}$. (*e*) $r = 1 - \cos\theta$.

2. Discuss the equations $r = a \cos n\theta$ and $r = a \sin n\theta$ for n an even integer; for n an odd integer. What is the difference in the form of the curve?

3. Discuss the equation $r = a \tan\theta$ and draw the corresponding locus.

4. Change the equation in Example X, Art. 76, to rectangular coördinates and compare with Art. 54. What is the locus?

5. Discuss the equation $r = \dfrac{b^2}{a - c\cos\theta}$, first when $c > a$ and then when $c < a$. What are the loci?

6. Discuss the equation $r = 2a \sin\theta \tan\theta$ and draw the locus. The curve is called the **cissoid of Diocles.**

7. Discuss the equation $r^2 = \dfrac{a^2}{\theta}$. The locus is the **lituus.**

8. Discuss the equation $r = a^\theta$. The locus is the **logarithmic spiral.**

TRANSFORMATION OF COÖRDINATES

77. Transformation of the axes. The equation of a given locus can be simplified often by changing the axes to a new position in the plane, and then finding the equation which the new coördinates of the points on the locus satisfy. The operation of changing the

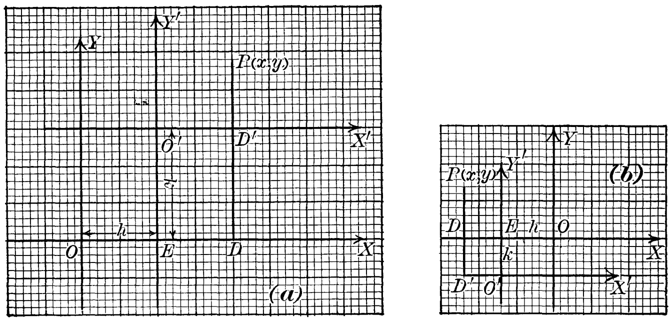

Fig. 58

axes is called **transformation of coördinates**, and the process of finding the new equation from the old is called **transformation of the equation.**

When the new axes $O'X'$ and $O'Y'$ are respectively parallel to the old axes OX and OY (Fig. 58), the transformation is called **translation of the axes**. Let the coördinates of the new origin O', referred to the old axes, be h and k; the coördinates of any point P, referred to the old axes, be x and y; and the coördinates of P, referred to the new axes, be x' and y'. Then, from either of the positions of O' shown in Fig. 58 (cf. Art. 3), we have

$$OD = OE + ED \quad \text{and} \quad DP = DD' + D'P$$
$$= h + x' \qquad\qquad\qquad = k + y'.$$

Hence, $$x = h + x' \quad \text{and} \quad y = k + y'. \qquad (1)$$

It is clear that equations (1) hold wherever the point O' may be situated, provided the new axes have the same direction as the old.

Example. Transform the equation $y = 2x + 3$ by translating the axes so that the new origin shall be the point (1, 5).

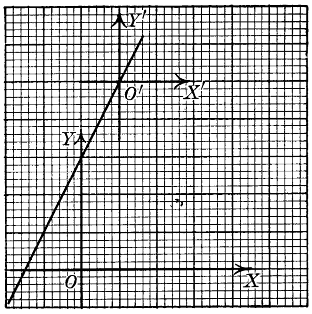

FIG. 59

Here, for any point (x, y) in the plane, and hence for any point on the locus of the given equation,

$$x = 1 + x' \quad \text{and} \quad y = 5 + y'.$$

Therefore, in terms of x' and y', the given equation becomes

$$5 + y' = 2(1 + x'), \text{ or } y' = 2x'. \quad (2)$$

The given equation $y = 2x + 3$ and the new equation $y' = 2x'$ represent the same locus; namely, the straight line shown in Fig. 59. The origin is, in the one case, at O, and in the other, at O'.

78. Rotation of the axes. When the origin is not moved, but the axes are each rotated through a given angle, the transformation is called **rotation of the axes.**

To obtain the equations for rotating the axes, let P be any point in the plane (Fig. 60) whose coördinates referred to the old axes are (x, y), and referred to the new axes are (x', y'). Let the angle XOX', through which the axes are rotated, be denoted by

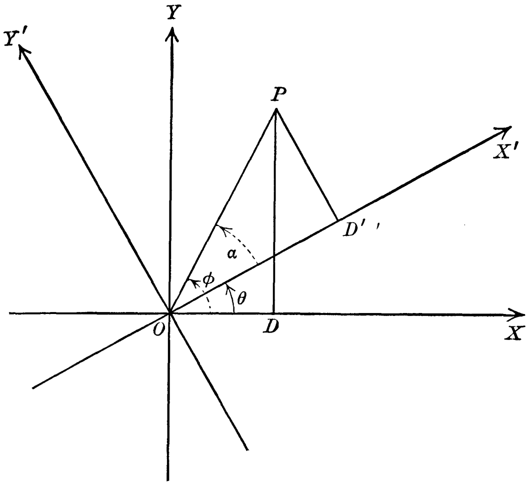

FIG. 60

θ, the angle XOP by ϕ, and the angle $X'OP$ by α. If $OP = r$, then

$$x = r\cos\phi = r\cos(\theta+\alpha) = r\cos\theta\cos\alpha - r\sin\theta\sin\alpha,$$
$$y = r\sin\phi = r\sin(\theta+\alpha) = r\sin\theta\cos\alpha + r\cos\theta\sin\alpha.$$

But $r\cos\alpha = x'$ and $r\sin\alpha = y'$; and therefore

$$\begin{aligned} \boldsymbol{x} &= \boldsymbol{x'\cos\theta - y'\sin\theta}, \\ \boldsymbol{y} &= \boldsymbol{x'\sin\theta + y'\cos\theta}. \end{aligned} \tag{1}$$

These equations express the old coördinates of any point in terms of the new coördinates. To obtain the new coördinates in terms of the old, we can solve equations (1) for x' and y', or we can derive x' and y' directly from the figure. In either way we find

$$\begin{aligned} x' &= x\cos\theta + y\sin\theta, \\ y' &= y\cos\theta - x\sin\theta. \end{aligned} \tag{2}$$

Example. Transform the equation $24xy - 7y^2 = 144$ by rotating the axes through the acute angle whose tangent is $\frac{3}{4}$.

Here $\sin\theta = \frac{3}{5}$ and $\cos\theta = \frac{4}{5}$, hence the equations for rotating the axes are

$$x = \tfrac{4}{5}x' - \tfrac{3}{5}y',$$
$$y = \tfrac{3}{5}x' + \tfrac{4}{5}y'.$$

Substituting in the given equation and reducing, we have

$$9\,x'^2 - 16\,y'^2 = 144, \text{ or } \frac{x'^2}{16} - \frac{y'^2}{9} = 1.$$

The given equation, therefore, represents an hyperbola whose semiaxes are 4 and 3.

EXERCISES

1. Transform the equation $3\,x + 7\,y = 8$ to a new set of axes parallel to the old set, and having the point $(4, -2)$ as origin.

2. Show that the equation $x^2 + y^2 = a^2$ is unaltered by rotating the axes through any angle θ. What is the geometrical interpretation of this fact?

3. Transform the equation $x^2 - y^2 = 10$ by rotating the axes through an angle of 45°.

4. Transform the equation $x^{\frac{1}{2}} + y^{\frac{1}{2}} = a^{\frac{1}{2}}$ by first translating the axes parallel to themselves, the new origin being at the point $\left(\frac{a}{4}, \frac{a}{4}\right)$, and then rotating the new axes through the angle 45°. What is the locus of the resulting equation? What is the locus of the original equation?

79. Removal of terms of first degree. When the given equation is an algebraic equation of the second degree in the variables and contains terms of the first degree, the latter can often be removed by translating the axes to a new origin, as illustrated by the following example.

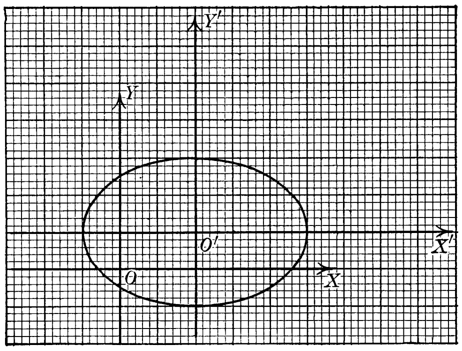

Fig. 61

Example. Given the equation $4\,x^2 + 9\,y^2 - 16\,x - 18\,y = 11$. Translate the axes so as to remove the terms of first degree.

Substituting for x and y their values in terms of x' and y', equations (1), Art. 77, we have

$$4(x' + h)^2 + 9(y' + k)^2 - 16(x' + h) - 18(y' + k) = 11.$$

The coefficients of x' and y' in this new equation are respectively $8h - 16$ and $18k - 18$. Hence, if we choose $h = 2$ and $k = 1$, the terms of first degree will drop out of the new equation and it reduces to

$$4x'^2 + 9y'^2 = 36.$$

This equation represents an ellipse whose semiaxes are 3 and 2, hence the given equation represents this ellipse. Figure 61 shows the curve and both sets of coördinate axes.

80. Removal of the term in xy. The term in xy can be removed from an equation of the second degree by rotating the axes through the proper angle. This is illustrated in Art. 78.

As another example, we will remove the term in xy from the equation

$$x^2 + 2xy + 2y^2 - 4 = 0.$$

Substituting the values of x and y from equations (1), Art. 78, and collecting terms, the given equation becomes

$$(\cos^2\theta + 2\sin^2\theta + 2\cos\theta\sin\theta)x'^2 + (2\cos\theta\sin\theta + 2\cos^2\theta - 2\sin^2\theta)x'y' + (\sin^2\theta + 2\cos^2\theta - 2\sin\theta\cos\theta)y'^2 - 4 = 0. \quad (1)$$

Putting the coefficient of $x'y'$ equal to zero, we have the equation

$$2\cos\theta\sin\theta + 2(\cos^2\theta - \sin^2\theta) = 0,$$

from which to determine θ. But this equation is equivalent to

$$\sin 2\theta + 2\cos 2\theta = 0 \text{ or } \tan 2\theta = -2.$$

Therefore $2\theta = \text{arc}\tan(-2) = 116° 34'$, nearly, or $\theta = 58° 17'$, nearly.

Since $$\tan 2\theta = -2,$$

we have $$\sin 2\theta = \frac{2}{\sqrt{5}},\ \cos 2\theta = \frac{-1}{\sqrt{5}}\cdot$$

Hence, $$\cos^2\theta = \frac{1 + \cos 2\theta}{2} = \frac{\sqrt{5} - 1}{2\sqrt{5}},$$

$$\sin^2\theta = \frac{1 - \cos 2\theta}{2} = \frac{\sqrt{5} + 1}{2\sqrt{5}},$$

and $$\sin\theta\cos\theta = \frac{\sin 2\theta}{2} = \frac{1}{\sqrt{5}}\cdot$$

Substituting these values in (1), it reduces to

$$\frac{x'^2}{3 - \sqrt{5}} + \frac{y'^2}{3 + \sqrt{5}} = 2.$$

Hence the given equation represents an ellipse.

EXERCISES

1. Remove the terms of first degree and then the term in xy from the equations

$$(a)\ xy - x - y = 0\ ;\ (b)\ xy - x + y = 0.$$

What are the loci which these equations represent?

2. Show that the terms of first degree cannot be removed from the equation

$$16\,x^2 - 24\,xy + 9\,y^2 - 20\,x - 110\,y + 225 = 0.$$

Try to generalize this result so as to tell at a glance whether the terms of first degree can be removed or not from any equation of the second degree.

3. Given the equation $x^3 - 3\,axy + y^3 = 0$. Rotate the axes through the angle 45° and compare the resulting equation with Example V, Art. 70. What locus does the given equation represent?

4. In exercise 2, rotate the axes through the angle $\theta = \text{arc}\tan\frac{4}{3}$ and then translate the axes, taking for new origin the point (2, 1). What locus does the equation represent?

81. Classification of algebraic curves.

THEOREM. *The degree of an algebraic equation in the variables x and y is unaltered by transformation of coördinates.*

For, in transforming the equation by translation or rotation of the axes we replace x and y by expressions of the first degree in x' and y' and therefore the degree of the equation cannot be raised by this process. Neither can it be lowered, for then it would be necessary to raise the degree in transforming back to the original axes, and we have just seen that the degree cannot be raised by a transformation of coördinates. Since the degree cannot be raised or lowered by a transformation of coördinates, it must remain unaltered.

Because of the theorem just proved, algebraic curves can be classified conveniently according to the degree of their equations, since we now know that the degree of the equation is independent of the position of the axes with reference to the curve.

If the degree of a given algebraic equation is any integer n, the corresponding curve is said to be of **order n.** The straight line is the only locus of order 1 (Art. 47). The circle, the ellipse, the hyperbola, and the parabola are all loci of order 2. The folium of Descartes is a locus of order 3 (Art. 70). The ovals of Cassini are loci of order 4 (Art. 54).

The succeeding chapters will be devoted to a special study of loci of orders 1 and 2.

MISCELLANEOUS EXERCISES

1. Show that the triangle whose vertices are $(3, 2)$, $(-1, -3)$, and $(-6, 1)$ is a right triangle.

2. On the line $y - 5 = 0$ a segment is laid off, having for abscissas of its extremities 2 and 5, and upon this segment an equilateral triangle is constructed. What are the coördinates of its third vertex?

3. Find the coördinates of the point dividing the segment $(5, 2)$ to $(4, -7)$ in the ratio $4:7$.

4. Change the polar equation $r = a + \frac{b}{\cos\theta}$ to one in rectangular coördinates. Plot the locus.

5. Remove the terms of first degree from the equation $4x^2+9y-8y-6=0$ and plot the resulting equation.

6. Find the rectangular equations of the asymptotes of the hyperbola whose polar equation is

$$r = \frac{7}{4\cos\theta - 3}.$$

7. Discuss the equation $y^2 = 4x^2 - x^3$ and plot the locus. Write the parametric equations of the locus if $y = tx$, t being the parameter.

8. Discuss the equation $y^2 = x^2\frac{2+x}{2-x}$ and plot the locus. Write the parametric equations if $y = tx$.

9. Write the parametric equations of the locus of $x^{\frac{1}{2}} + y^{\frac{1}{2}} = a^{\frac{1}{2}}$, assuming $x = a\cos^4\theta$.

10. OB is the crank of an engine and AB the connecting rod, A being the piston. A moves in a straight line passing through O. Find the equation of the locus of any point P on the connecting rod. Let P be a units from A and b units from B, and let r be the length of the crank, OB. Discuss the equation and plot the locus. Write the parametric equations of the locus, assuming $y = a\sin\theta$. What is the locus when $r = a + b$?

11. Discuss the equation $y^2(a - x) - x^2(a + x) = 0$, and plot the locus. The curve is called the **strophoid**.

12. Construct the locus of $y = \sin 2x + \frac{x^2}{10}$; of $y = e^{-x} + 4x^2$; of $y = \frac{1}{x}\cdot\cos x$.

13. Discuss the equation $r^2 = a^2\sin 2\theta$ and plot the locus.

14. Discuss the equation $r^2 = a^2\tan\theta$ and plot the locus.

CHAPTER VI

LOCI OF FIRST ORDER

82. Linear equations. We have seen (Art. 47) that the equation of every straight line is of the first degree in x and y; and conversely, that every equation of first degree in x and y is the equation of a straight line. For this reason, an equation of the first degree in x and y is called a **linear equation.**

Every linear equation is of the form

$$Ax + By + C = 0, \tag{1}$$

where A, B, and C are constants. These constants can have any values, except that A and B cannot both be zero, for then the equation would contain neither variable. If A is zero, (1) is the equation of a line parallel to the X-axis, for then y has the value $-\frac{C}{B}$ for all values of x. If B is zero, (1) is the equation of a line parallel to the Y-axis, for then x has the value $-\frac{C}{A}$ for all values of y. Finally, if C is zero, (1) is the equation of a line passing through the origin, for then the equation is satisfied when $x = 0$ and $y = 0$.

For A, B, C different from zero, the slope of the line is given by the formula

$$\boldsymbol{m = -\frac{A}{B},}$$

and the intercepts a, b on the X- and Y-axes by

$$\boldsymbol{a = -\frac{C}{A}} \text{ and } \boldsymbol{b = -\frac{C}{B},}$$

respectively.

EXERCISES

1. Find the slopes and intercepts of the lines whose equations are the following:

(*a*) $x + \sqrt{3}\,y + 10 = 0.$
(*b*) $y = x - 6.$
(*c*) $5x - 12y = 13.$
(*d*) $2x - 3y = 4.$

(e) $x - a = 0$. (f) $4y + 3x = 24$.
(g) $5x + 4y = 20$. (h) $2x - 4y + 9 = 0$.
(i) $2x + 3y = 0$. (j) $y = 4$.
(k) $Ax + By + C = 0$. (l) $(a^2 - b^2)x = (a + b)y + c$.

2. If a and b represent the intercepts on the X- and Y-axes respectively, and m the slope, determine the equations of the lines for which
(1) $a = 2$, $b = -3$. (2) $a = -1$, $m = 4$.
(3) $b = 3$, $m = -2$. (4) $m = -5$, $a = -2$.
(5) $a = 2$, and passing through the point $(4, -3)$.
(6) Passing through the points $(-1, 2)$ and $(5, -4)$.
(7) $a = \frac{B}{C}$, $m = -\frac{A}{C}$.

83. Intersection of two lines. The coördinates of the point of intersection of two lines must satisfy the equation of each line, since the point lies on each line. To find the coördinates of the point of intersection it is only necessary, therefore, to solve the equations simultaneously for x and y. For example, $x = 3$ and $y = 4$ is the common solution of the two equations $x - y + 1 = 0$ and $4x + y - 16 = 0$; and these are the equations of two straight lines which intersect in the point (3, 4).

In general, two straight lines intersect in one and only one point. But they may be:

(1) Parallel to each other.

(2) Coincident.

In the first case the slopes of the lines are equal and their equations have no common solution. For example, the equations

$$2x - 3y = 4 \text{ and } 2x - 3y = 7 \qquad (1)$$

are the equations of a pair of parallel lines, since the slope of each is $\frac{2}{3}$. The equations have no common solution. The equations of a pair of parallel lines are called *incompatible* or *inconsistent*.* Thus equations (1) are incompatible. Obviously $2x - 3y$ cannot be 4 and 7 at the same time for any values of x and y.

In the second case the slopes of the lines are also equal, but their equations have an indefinite number of common solutions, since any pair of values of x and y that satisfies one equation must also satisfy the other. The equations, therefore, can differ only by a constant factor. The equations of a pair of coincident lines are called **dependent.** Thus, the equations

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* See Rietz and Crathorne, *College Algebra*, p. 49.

$$2x - 3y = 4 \text{ and } 4x - 6y = 8$$

are dependent and are the equations of a pair of coincident lines.

EXERCISES

1. Find the intersections of the lines represented by the following pairs of equations. Tell which are inconsistent and which are dependent equations.
(*a*) $2x + 3y = 12$, $4x - y = 5$. (*b*) $3x + 5y = 1$, $6x + 10y + 7 = 0$.
(*c*) $5x - 2y + 4 = 0$, $x - .4y = -.8$. (*d*) $x + 3y = 15$, $3x - y = 5$.
Draw the lines in each case.

2. Write an equation representing the same straight line as $5x+4y-20=0$ in which the sum of the coefficients shall be 22; in which the product of the first and third coefficients shall be equal to the second.

3. Change the equation $3x - 4y + 12 = 0$ into another representing the same straight line and having the square of the second coefficient equal to twice the third minus four times the first; having the product of all three coefficients equal to minus three times the last.

4. The equations $5x - 2y - 3 = 0$ and $Ax + By + C = 0$ are dependent and $B^2 + 2(A + C) = 24$. Find A, B, and C.

84. The pencil of lines. *Let $u = 0$ and $v = 0$ be the equations of two straight lines, then the equation*

$$u + kv = 0 \qquad (1)$$

is the equation of a straight line passing through the intersection of $u = 0$ and $v = 0$, whatever value is given to k.

Here u and v are each expressions of the first degree in x and y and therefore $u + kv = 0$ is the equation of some straight line (Art. 47). Moreover, $u + kv = 0$ is satisfied by the coördinates of the point of intersection of $u = 0$ and $v = 0$ and therefore it is the equation of a straight line passing through the intersection of $u = 0$ and $v = 0$.

If k is allowed to vary, a series of lines is obtained each passing through the intersection of $u = 0$ and $v = 0$. The totality of lines so obtained is called a **pencil of lines** (Fig. 62).

The constant k can be determined so that the line $u + kv = 0$ shall satisfy any single condition, such as passing through a given point, having a given slope, etc.

Example. Find the equation of the line passing through the intersection of $2x + 3y - 4 = 0$ and $x + 2y - 5 = 0$ and also through the point (2, 3).

The line whose equation is required is one of the pencil

$$2x + 3y - 4 + k(x + 2y - 5) = 0. \quad (2)$$

Since the line is to pass through the point (2, 3), these coördinates must satisfy the equation. Hence $k = -3$. Substituting this value of k in (2), we have the required equation,

$$x + 3y - 11 = 0.$$

This result may be verified by solving the given equations simultaneously and then finding the equation of the line passing through the common point and the point (2, 3) in the usual way.*

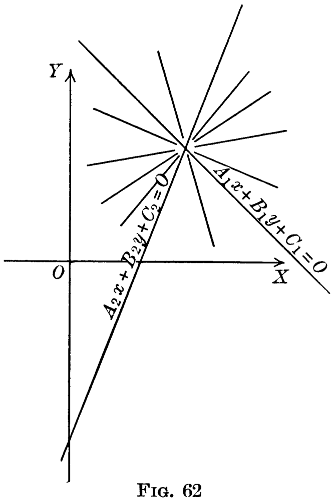

Fig. 62

EXERCISES

1. What is the equation of the line passing through the origin and through the intersection of the lines $x + 3y - 8 = 0$ and $4x - 5y = 10$?

2. The equations of the sides of a triangle are $5x-6y=16$, $4x+5y=20$, and $x + 2y = 0$. Find the equations of the lines passing through the vertices and parallel to the opposite sides.

3. Find the equation of the line which passes through the intersection of the lines $2x - 3y + 1 = 0$ and $x + 5y + 6 = 0$ and is perpendicular to the first of these lines. Which is parallel to the line $5x - y + 10 = 0$.

4. What is the equation of the line which passes through the intersection of the lines $y = 7x - 4$ and $y = -2x + 5$ and makes an angle of 60° with the positive end of the x-axis?

5. Find the equation of the line which passes through the intersection of the lines $5y - 2x - 10 = 0$ and $y + 4x - 3 = 0$, and also through the intersection of the lines $10y + x + 21 = 0$ and $3y - 5x + 1 = 0$.

Suggestion. The equations

$$5y - 2x - 10 + k(y + 4x - 3) = 0 \text{ and } 10y + x + 21 + k'(3y - 5x + 1) = 0$$

must be *dependent* (Art. 83).

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* The theorem of this article holds when u and v are expressions of any degree in x and y. $u + kv = 0$ is then the equation of a **pencil of curves**.

85. The pair of lines. *Let $u = 0$ and $v = 0$ be the equations of two straight lines, then the locus of the equation*

$$u \cdot v = 0$$

is the pair of lines $u = 0$ and $v = 0$ taken together.

For the equation $u \cdot v = 0$ is satisfied only when one of the factors, u or v, is equal to zero or when both factors are equal to zero. But the two straight lines pass through all the points whose coördinates satisfy the equations $u = 0$ and $v = 0$, and through no others. Consequently these lines, taken together, form the locus of the equation $u \cdot v = 0$ (Art. 63).*

Example. The locus of the equation $x^2 - y^2 = 0$ is the pair of lines $x + y = 0$ and $x - y = 0$.

EXERCISES

1. Draw the pairs of lines whose equations are the following:

(*a*) $x^2 + xy = 0$. (*b*) $2x^2 + 5xy - 3y^2 = 0$. (*c*) $x^2 - 5x + 6 = 0$.
(*d*) $2y^2 - xy + 4x - 9y + 4 = 0$. (*e*) $x^2 - y^2 - 2y - 1 = 0$.

2. Write the equation of the pair of lines each of which passes through the origin and whose slopes are respectively $\sqrt{3}$ and $-\sqrt{3}$.

3. Write the equation of the pair of lines each of which passes through the point (1, 2) and whose slopes are respectively 2 and $-\frac{1}{2}$.

86. The normal form. We have seen (Art. 57) that the polar equation of a straight line is

$$r \cos(\theta - \alpha) = p,$$

where p is the length of the perpendicular from the origin on the line and α is the inclination of this perpendicular to the X-axis.

Expanding $\cos(\theta - \alpha)$, the equation becomes

$$r(\cos\theta \cos\alpha + \sin\theta \sin\alpha) = p.$$

Since $r\cos\theta = x$ and $r\sin\theta = y$, the equation in rectangular coördinates is

$$\boldsymbol{x \cos \alpha + y \sin \alpha = p.} \tag{1}$$

---

* The theorem of this article holds when u and v are expressions of any degree in x and y. The locus of $u \cdot v = 0$ is then called a **composite curve.**

This is called the **normal form** of the equation of a straight line (Fig. 63).

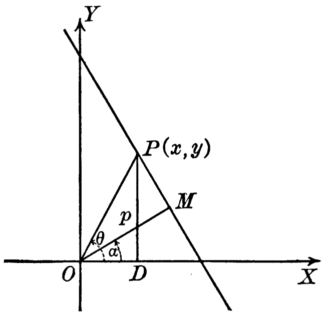

Fig. 63

87. Reduction of $Ax+By+C=0$ to the normal form. The problem before us consists in reducing the given equation

$$Ax + By + C = 0 \tag{1}$$

to the form

$$x \cos \alpha + y \sin \alpha - p = 0. \tag{2}$$

Since (2) is to be the equation of the same line as (1), the two equations can differ only by a constant factor (Art. 83). Hence we must have

$$\cos \alpha = kA, \ \sin \alpha = kB, \text{ and } -p = kC, \tag{3}$$

where k is the constant factor. From the first two of these equations, by squaring and adding, we get

$$1 = k^2(A^2 + B^2), \text{ or } k = \frac{1}{\pm\sqrt{A^2 + B^2}}.$$

Therefore

$$\cos \alpha = \frac{A}{\pm\sqrt{A^2+B^2}}, \ \sin \alpha = \frac{B}{\pm\sqrt{A^2+B^2}}, \text{ and } -p = \frac{C}{\pm\sqrt{A^2+B^2}}. \tag{4}$$

In order to determine which sign shall be given to the radical in any numerical example, we shall assume that *p is always a positive number* and then, from (4), *the sign of the radical must be opposite to the sign of C.*

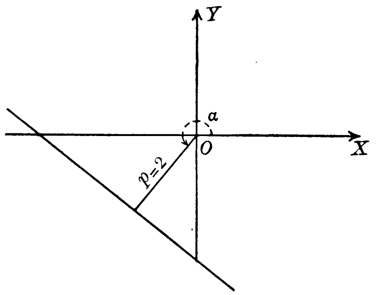

Fig. 64

For example, to reduce $3x+4y+10=0$ to the normal form, we divide both members of the equation by $-\sqrt{9+16} = -5$ and obtain

$$-\tfrac{3}{5}x - \tfrac{4}{5}y - 2 = 0.$$

Therefore

$$p = 2, \ \cos \alpha = -\tfrac{3}{5}, \ \sin \alpha = -\tfrac{4}{5}, \text{ and } \alpha = 233° \, 8'$$

(Fig. 64).

EXERCISES

1. Write the equations of the lines for which:

(*a*) $p = 5,\ \alpha = 60°$. (*b*) $p = 5,\ \alpha = 120°$. (*c*) $p = -5,\ \alpha = 330°$.
(*d*) $p = 0,\ \alpha = 225°$. (*e*) $p = 1,\ \alpha = 45°$. (*f*) $p = 6,\ \alpha = -60°$.

2. Reduce the following equations to the normal form and plot the lines of which they are the equations:

(*a*) $4x - 3y = 25$. (*b*) $x + 4 = 0$. (*c*) $x + 2y = -8$.
(*d*) $5y - 3 = 0$. (*e*) $2x - y = 0$. (*f*) $x - 3y + 4 = 0$.

3. What system of lines is given by $x \cos \alpha + y \sin \alpha - p = 0$ when α is constant and p varies? When p is constant and α varies?

4. Two lines can be drawn through the point (2, 5) and tangent to the circle $x^2 + y^2 = 25$. Find the equation of each line. Draw the figure.

88. Distance from a line to a point. With the help of the normal form of the equation of a straight line, it is easy to find the length of the perpendicular DP drawn from a given line AB to a given point $P(x_1, y_1)$ (Fig. 65). Thus, let the equation of AB, reduced to the normal form, be

$$x \cos \alpha + y \sin \alpha - p = 0. \quad (1)$$

Through P draw RS parallel to AB, and let $OM = p_1$ be the length of the perpendicular from O to RS. Then the normal form of the equation of RS is

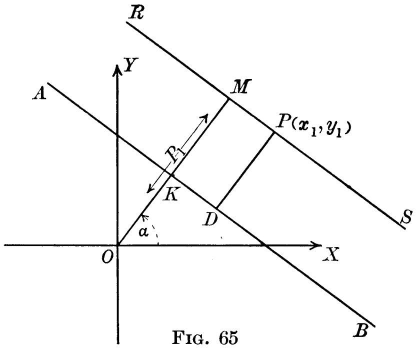

FIG. 65

$$x \cos \alpha + y \sin \alpha - p_1 = 0. \quad (2)$$

Since P is on RS, the coördinates x_1 and y_1 must satisfy (2). Hence,

$$x_1 \cos \alpha + y_1 \sin \alpha = p_1. \quad (3)$$

Subtracting p from both members of (3), we have

$$x_1 \cos \alpha + y_1 \sin \alpha - p = p_1 - p = OM - OK = DP.$$

Hence the following rule:

To find the distance from a line to a point, reduce the equation of the given line to the normal form with the right member equal to zero; substitute the coördinates of the given point in the left member. The result is the required distance.

The sign of the result will be negative when $p_1 < p$; that is, when the given point and the origin are on the same side of the given line. The sign will be positive in the contrary case.

Example. Find the distance from the line $3x + 4y + 12 = 0$ to the point (2, 3).

Here the normal form of the given equation is

$$\frac{3x + 4y + 12}{-5} = 0.$$

Substituting 2 and 3 for x and y in the left member, we find the required distance is -6. Figure 66 illustrates the example. Note that the given point and the origin are on the same side of the line.

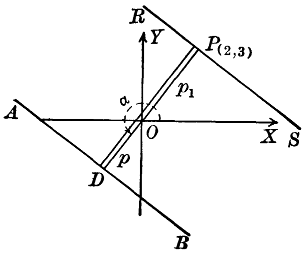

FIG. 66

EXERCISES

1. Find the distance of the point (3, 5) from the line $2x - 3y + 6 = 0$.

2. Find the distance between the parallel lines $7x - 8y = 15$ and $7x - 8y = 40$.

3. The line $5x + 12y = 25$ touches a circle whose center is the origin. Find the radius of the circle and write its equation.

4. The line $6x - 8y = 15$ touches a circle whose center is the point $(-3, 4)$. Find the radius of the circle and write its equation.

5. Find the equations of the circles inscribed in the following triangles: —

(*a*) $x + 2y - 5 = 0,\ 2x - y - 5 = 0,\ 2x + y + 5 = 0.$
(*b*) $3x + y - 1 = 0,\ x - 3y - 3 = 0,\ x + 3y + 11 = 0.$
(*c*) $x + 2 = 0,\ y - 3 = 0,\ x + y = 0.$
(*d*) $x = 0,\ y = 0,\ x + y + 3 = 0.$

89. The angle which one line makes with another. When the equations of two lines are given, the slopes of the lines are known. The tangent of the angle which one line makes with the other can then be computed as in Art. 13. For example, we will find the angle which the line $x + 2y - 3 = 0$ makes with the line $3x - y + 4 = 0$. Here the slope of the first line is $m_1 = -\frac{1}{2}$ and the slope of the second line is $m_2 = 3$. Therefore, by formula (4), Art. 13, we have

$$\tan\phi = \frac{m_2 - m_1}{1 + m_1 m_2} = \frac{3 + \frac{1}{2}}{1 - \frac{3}{2}} = -7.$$

The angle ϕ is approximately 98° 8′. The student should construct the figure to illustrate this example.

EXERCISES

1. Find the angle which the line $3x - y + 2 = 0$ makes with the line $2x + y - 2 = 0$.

2. Find the angle which the line $2x - 5y + 1 = 0$ makes with the line $x - 2y + 3 = 0$.

3. Find the angles of the triangle formed by the lines $x + 3y - 4 = 0$, $3x - 2y + 1 = 0$, and $x - y + 3 = 0$. Draw the figure.

4. Find the equations of the bisectors of the angles formed by the two lines $3x - 4y + 2 = 0$ and $4x - 3y - 1 = 0$. Show that the bisectors are perpendicular to each other.

SUGGESTION. Any point on the bisector of an angle is equidistant from the lines forming the angle.

5. Find the equations of the bisectors of the interior angles of the triangles formed by the lines $5x - 12y = 0$, $5x + 12y + 60 = 0$, and $12x - 5y - 60 = 0$. Show that the bisectors meet in a point.

6. Generalize the preceding exercise and thus show that the bisectors of the interior angles of any triangle meet in a point. Choose the coördinate axes so that the equations of the sides of the triangle are as simple as possible.

7. Show that, for any straight line,

$$\cos\alpha = \frac{p}{a},\ \sin\alpha = \frac{p}{b},\ \text{and}\ \tan\alpha = -\frac{1}{m},$$

where a and b are respectively the X- and Y-intercepts, m is the slope, p is the perpendicular from the origin on the line, and α is the inclination of this perpendicular to the X-axis.

8. Write the equation of the straight line for which:

(1) $a = 3$, $b = -2$; (2) $a = 5$, $p = 3$; (3) $m = \frac{2}{3}$, $p = 5$.

9. Of the five numbers a, b, m, p, and α, having given any two, the other three can, in general, be determined. What cases form an exception to this general rule? From the given pairs in exercise 8, determine the other three.

CHAPTER VII

LOCI OF SECOND ORDER. EQUATIONS IN STANDARD FORM

DIRECTRICES

90. Review. We have found that the circle, the ellipse, the hyperbola, and the parabola are curves of the second order (Art. 81). The definitions of these curves and also the process of finding the standard forms of their equations should be reviewed (Chapter IV).

In this chapter we shall derive some important properties of these curves, making use of the standard forms of the equations.

EXERCISES

1. If a and b represent the lengths of the semiaxes and e the eccentricity write the standard equation of the ellipse for which ; —

(1) $a = 3$ and $b = 2$.
(2) $b = 3$ and $e = \frac{1}{2}$.
(3) $a = 6$ and $e = \frac{2}{3}$.
(4) $b = 4$ and $c = ae = 3$.
(5) $a = 5$ and $c = 3$.
(6) $c = 4$ and $e = \frac{1}{3}$.

2. Find a, b, c, and e from the following equations : —

(1) $x^2 - 25y^2 = 25$.
(2) $x^2 + 25y^2 = 25$.
(3) $9x^2 - 4y^2 = 36$.
(4) $9x^2 + 4y^2 = 36$.
(5) $2x^2 - 5y^2 = 20$.
(6) $2x^2 + 5y^2 = 20$.

3. Find the lengths of the focal radii of the ellipse $x^2 + 9y^2 = 18$, drawn to the points whose abscissa is -2.

4. Find the lengths of the focal radii of the hyperbola $9x^2 - 4y^2 = 65$, drawn to the points whose ordinate is 2.

5. Show that the circle is the limiting form of the ellipse as a and b approach equality. What is the eccentricity of the circle and where are its foci ?

6. When the semiaxes of an hyperbola are equal, the hyperbola is called **equilateral.** Show that the distance from any point on an equilateral hyperbola to the center of the curve is a mean proportional between the focal radii drawn to the same point.

91. Directrices. Let e represent the eccentricity and a the semitransverse axis of an ellipse or an hyperbola. The lines drawn perpendicular to the transverse axis, one at the distance $\frac{a}{e}$ to the right, and the other $\frac{a}{e}$ to the left of the center, are called the **directrices.** In Figs. 67 and 68, the lines DE and D_1E_1 are the directrices.

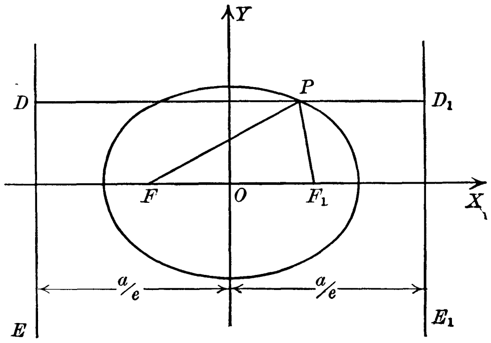

FIG. 67

The focus and directrix on the same side of the center are said to correspond to each other. Thus, DE corresponds to F and D_1E_1 to F_1.

92. A fundamental theorem. *If the length of the focal radius to any point on an ellipse or an hyperbola is divided by the distance of the point from the corresponding directrix, then the ratio so formed is constant and equal to the eccentricity of the curve.*

We are to prove (Figs. 67 and 68) that

$$\frac{FP}{PD}=\frac{F_1P}{PD_1}=e,$$

where P is any point on either curve. From Art. 50, we have $FP = a + ex$ and $F_1P = a - ex$, and from Fig. 67,

$$PD=\frac{a}{e}+x \text{ and } PD_1=\frac{a}{e}-x.$$

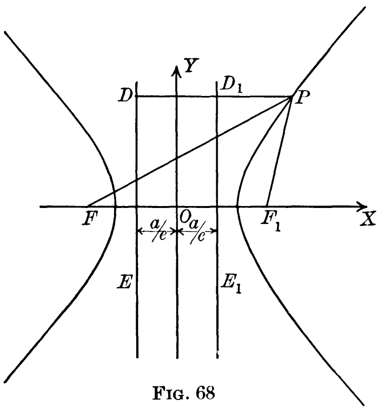

FIG. 68

Hence, $$\frac{FP}{PD}=\frac{a+ex}{\frac{a}{e}+x}=e \text{ and } \frac{F_1P}{PD_1}=\frac{a-ex}{\frac{a}{e}-x}=e.$$

The theorem is proved for the hyperbola in the same way, making use of the lengths of the focal radii (Art. 52) and Fig. 68.

93. Construction of an ellipse or an hyperbola. The theorem in the preceding article furnishes a convenient method for constructing an ellipse or an hyperbola when the eccentricity and the distance from a focus to the corresponding directrix are known. For example, to construct the hyperbola whose eccentricity is $\frac{3}{2}$ and the distance from one focus to the corresponding directrix is 2, let F be one focus and AB the corresponding directrix (Fig. 69), so that the distance QF is 2. Draw a parallel through F to AB, and lay off the equal distances FM and FM' so that*

$$\frac{FM}{QF}=\frac{FM'}{QF}=\frac{3}{2}.$$

Draw QM and QM' and a series of parallels to AB. Let one of these parallels meet QM and QM' in L and L', respectively, and QF in D. The triangles QFM and QDL are similar, and therefore

$$FM : QF :: DL : QD,$$

or the ratio $DL : QD$ is equal to the given eccentricity $\frac{3}{2}$. With F as center and DL as radius, draw arcs of a circle cutting LL' in R and R'. Then R and R' are points on the curve, since $FR : QD = DL : QD = 3 : 2$. In this way as many points of the curve can be located as may be desired.

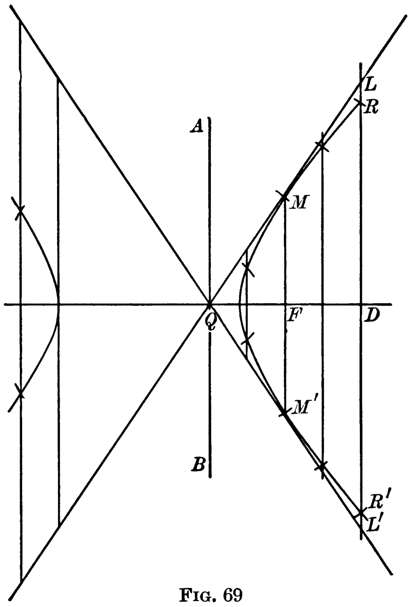

FIG. 69

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* Here, and for the most part throughout this chapter, we are not concerned with directed segments or with directed angles.

An ellipse can be constructed in a similar way.

By construction, the angle $FQM =$ arc tan e and is, therefore, greater than 45° for the hyperbola and less than 45° for the ellipse (Arts. 50 and 52).

94. Two common properties. By definition (Art. 53), the length of the focal radius to any point on a parabola divided by the distance from the point to the directrix is equal to unity. Comparing this statement of the definition of the parabola with the theorem in Art. 92, we are led to define the eccentricity of the parabola as unity. Consequently, the ellipse, hyperbola, and parabola have the following important property:

(A) *If the length of the focal radius to any point on one of these curves is divided by the distance from the point to the corresponding directrix, then the ratio so formed is constant and equal to the eccentricity of the curve; this ratio is less than* 1 *for the ellipse, equal to* 1 *for the parabola, and greater than* 1 *for the hyperbola.*

The three curves have another important property in common; namely,

(B) *They are sections of a right circular cone* (Part II, Art. 159).

The circle is also a section of a right circular cone. The four curves are therefore called **conic sections**, or more briefly, **conics**.

Note. The conics were originally studied by the Greeks, who used property (B) as a definition. Property (A) was probably known to Euclid and his contemporaries (300 B.C.), but the earliest mention of it now known to exist occurs in the "Collections of Pappus" (100 A.D.).

EXERCISES

1. In an ellipse, given $a = 3$ and $b = 2$. Find c and e, locate accurately the foci and directrices, find the distance from a focus to the corresponding directrix; write the standard equation in rectangular coördinates, the polar equation with the pole at the left-hand focus, and the parametric equations.

2. Find a and b for the hyperbola constructed in Art. 93. Write the standard equation in rectangular coördinates, the polar equation with the pole at the left-hand focus, and the parametric equations.

3. Make an accurate construction of the ellipse for which $e = \frac{1}{2}$ and $a = 6$. Locate the foci and the directrices and write the three standard forms of its equation as in the preceding exercises.

4. Construct the hyperbola for which $a = 4$ and $b = 5$. Locate the foci and directrices and write the three standard forms of its equation.

5. The length of the focal radius drawn to one extremity of the minor axis of an ellipse is 5 and the eccentricity is $\frac{4}{5}$. Construct the curve and locate the foci and directrices.

TANGENTS

95. Equation of a tangent in terms of the slope. If a straight line meets a curve in the points P and Q, these points will move along the curve when the line is either rotated about some point in the plane or moved parallel to itself. If it is possible to pass continuously along the curve from P to Q, it will be possible to move the line in either of the ways mentioned so as to cause P and Q to coincide in a point R. The line PQ is then **tangent** to the curve at R, and R is the **point of contact**.

To find the coördinates of the points of intersection of a curve with a straight line, it is necessary to solve the equation of the curve and the equation of the line simultaneously.

The circle. Let the given curve be the circle

$$x^2 + y^2 = a^2, \tag{1}$$

and take the equation of the line in the slope form

$$y = mx + k. \tag{2}$$

Eliminating y between (1) and (2), we see that the x-coördinates of the points of intersection are the roots of the equation

$$(1 + m^2)x^2 + 2mkx + (k^2 - a^2) = 0. \tag{A}$$

The line will move parallel to itself when k is allowed to vary, and the points of intersection, P and Q (Fig. 70), will coincide when, and only when, the roots of equation (A) are equal; that is, when

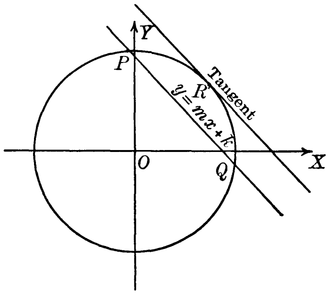

Fig. 70

$$4(1 + m^2)(k^2 - a^2) = 4m^2k^2.$$

Solving this equation for k, we have

$$k = \pm a\sqrt{1+m^2}.$$

Hence, when k has either of these values, the roots of (A) are equal and the points of intersection of the circle with the line coincide. Therefore, *the lines*

$$\boldsymbol{y = mx \pm a\sqrt{1+m^2}} \tag{3}$$

are tangents to the circle $x^2+y^2=a^2$.

The ellipse. Similarly, solving the equation of the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \tag{4}$$

and equation (2) simultaneously, we find that the x-coördinates of the points of intersection are the roots of the equation

$$\boldsymbol{(a^2m^2+b^2)x^2+2\,a^2mkx+a^2(k^2-b^2)=0.} \tag{B}$$

The roots will be equal, and consequently the line a tangent to the ellipse, when

$$4\,a^2(a^2m^2+b^2)(k^2-b^2)=4\,a^4m^2k^2, \text{ or } k=\pm\sqrt{a^2m^2+b^2}.$$

Therefore, *the lines*

$$\boldsymbol{y = mx \pm \sqrt{a^2m^2+b^2}} \tag{5}$$

are tangents to the ellipse whose equation is given in (4).

The hyperbola. The x-coördinates of the points of intersection of the hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1 \tag{6}$$

with the line $y=mx+k$ are the roots of the equation

$$\boldsymbol{(a^2m^2-b^2)x^2+2\,a^2mkx+a^2(k^2+b^2)=0,} \tag{C}$$

and these roots are equal when k has either of the values

$$k=\pm\sqrt{a^2m^2-b^2}.$$

Therefore, *the lines*

$$\boldsymbol{y = mx \pm \sqrt{a^2m^2-b^2},} \tag{7}$$

are tangents to the hyperbola whose equation is given in (6).

The parabola. The x-coördinates of the points of intersection of the parabola

$$y^2=4\,px \tag{8}$$

with the line $y = mx + k$ are the roots of the equation

$$\boldsymbol{m^2x^2 + (2\,mk - 4\,p)x + k^2 = 0,} \tag{D}$$

and these roots are equal when k has the value $\dfrac{p}{m}$.

Therefore, *the line*

$$\boldsymbol{y = mx + \frac{p}{m}} \tag{9}$$

is a tangent to the parabola $y^2 = 4\,px$.

As an example of the use of the foregoing formulas, we shall find the equations of the tangents to the hyperbola

$$x^2 - 4\,y^2 = 36$$

which have the slope $\frac{5}{6}$. Here $a = 6$, $b = 3$, and $m = \frac{5}{6}$. Substituting in (7), we find the required equations are

$$6\,y = 5\,x \pm 24.$$

Again, to find the equations of the tangents to the ellipse

$$4\,x^2 + 9\,y^2 = 36,$$

which pass through the point (2, 3), use equation (5). Here $a = 3$, $b = 2$, and we are to find m so that the tangents pass through the given point. Hence we must have

$$3 = 2\,m \pm \sqrt{9\,m^2 + 4},$$

from which we find

$$m = \frac{-6 \pm \sqrt{61}}{5}.$$

The required equations are therefore

$$y - 3 = \frac{-6 \pm \sqrt{61}}{5}(x - 2).$$

The formulas (A), (B), (C), and (D) are of frequent use in what follows.

Note that properties of the hyperbola, expressed by equations involving the semiaxes, can be derived from the corresponding properties of the ellipse by changing the sign of b^2. Thus, equations (B) and (C) differ only in the sign of b^2.

EXERCISES

1. Find the equations of the tangents to the following conics:

(*a*) $y^2 = 4\,x$, slope $= \frac{1}{2}$.

(*b*) $x^2 + y^2 = 16$, slope $= -\frac{4}{3}$.

(*c*) $9\,x^2 + 16\,y^2 = 144$, slope $= -\frac{1}{4}$.

(*d*) $x^2 - 4\,y^2 = 36$, passing through the point (3, 4).

(*e*) $x^2 + 4\,y^2 = 36$, perpendicular to $6\,x - 4\,y + 9 = 0$.

2. Find the equations of the common tangents to the following pairs of conics. Construct the figures.

(*a*) $y^2 = 5x$ and $9x^2 + 9y^2 = 16$.
(*b*) $9x^2 + 16y^2 = 144$ and $7x^2 - 32y^2 = 224$.
(*c*) $x^2 + y^2 = 49$ and $13x^2 + 50y^2 = 650$.

SUGGESTION. Find the equations of tangents to each conic in terms of the slope and then determine the slope so that the two equations shall be dependent (Art. 83).

3. Prove that two tangents to the parabola $y^2 = 4px$ which are perpendicular to each other intersect on the directrix.

SUGGESTION. The slopes of the tangents are negative reciprocals of each other. Hence their equations are

$$y = mx + \frac{p}{m} \text{ and } y = -\frac{x}{m} - pm.$$

But these lines intersect in a point whose abscissa is $-p$, whatever the value of m. Construct a figure illustrating this exercise.

4. Prove that two tangents to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ which are perpendicular to each other intersect upon the circle $x^2 + y^2 = a^2 + b^2$.

SUGGESTION. The equations of the two tangents are

$$y = mx + \sqrt{a^2m^2 + b^2} \text{ and } y = -\frac{x}{m} + \left(\frac{\sqrt{a^2 + b^2m^2}}{m}\right).$$

The point of intersection must satisfy both these equations; hence the equation of its locus is found by eliminating m from these equations. To do this, remove the radicals from each equation by transposition and squaring. Thus,

$$(y - mx)^2 = a^2m^2 + b^2 \text{ and } (my + x)^2 = a^2 + b^2m^2.$$

Add these equations, member to member, and divide by the common factor $1 + m^2$. The circle $x^2 + y^2 = a^2 + b^2$ is called the **director circle for the ellipse**. Construct a figure illustrating this exercise.

5. The locus of the intersection of a pair of perpendicular tangents to the hyperbola is called the **director circle for the hyperbola.** Find its equation and show that it is a real circle only when $a > b$; it reduces to a point for the equilateral hyperbola, $a = b$; and is imaginary for $a < b$. Construct a figure illustrating this exercise.

6. If a perpendicular is dropped from either focus of an ellipse (or an hyperbola) upon a tangent, show that the locus of its intersection with the tangent is a circle whose center coincides with the center of the curve and whose diameter is the transverse axis of the curve.

SUGGESTION. For the ellipse, the equations of the tangent and the perpendicular through the left-hand focus are, respectively,

$$y = mx + \sqrt{a^2m^2 + b^2} \text{ and } y = -\left(\frac{x + c}{m}\right).$$

Hence, $$(y - mx)^2 = a^2m^2 + b^2,$$
and $$(my + x)^2 = c^2 = a^2 - b^2.$$
Adding and dividing by the common factor $1 + m^2$, we have

$$x^2 + y^2 = a^2.$$

This circle is called the **major auxiliary circle**. The equation of the **minor auxiliary circle** is $x^2 + y^2 = b^2$ (cf. Art. 61). Construct a figure illustrating this exercise.

7. Show that the locus of the intersection of a tangent to $y^2 = 4px$ with the perpendicular from the focus is the Y-axis.

8. Show that the product of the perpendiculars from the foci upon any tangent to an ellipse is constant and equal to b^2. State and prove the corresponding property for the hyperbola.

96. Coördinates of the point of contact. Equations (A), (B), (C), and (D) of the preceding article serve to find the coördinates of the point of contact on a tangent having the given slope m. Thus, for the ellipse,

$$(a^2m^2 + b^2)x^2 + 2\,a^2mkx + a^2(k^2 - b^2) = 0 \text{ and } k^2 = a^2m^2 + b^2.$$

Therefore, $$k^2x^2 + 2\,a^2mkx + a^4m^2 = 0.$$

The left member of this equation is a perfect square, as it should be, and gives for the x-coördinate of the point of contact

$$x = \frac{-a^2m}{k}.$$

Since the point of contact is on the line $y = mx + k$, we have

$$y = mx + k = \frac{-a^2m^2}{k} + k = \frac{-a^2m^2 + k^2}{k} = \frac{b^2}{k}.$$

Therefore, *the points of contact on the tangents having the slope* m *are* $\left(\frac{-a^2m}{k}, \frac{b^2}{k}\right)$, *where* k *has the values* $\pm\sqrt{a^2m^2 + b^2}$.

Similarly, for the parabola, making use of equation (D) and the corresponding value of k, the x-coördinate of the point of contact is given by the equation

$$m^2x^2 - 2\,px + \frac{p^2}{m^2} = 0,$$

from which $x=\frac{p}{m^2}$. Substituting in $y=mx+k$, we find that $y=\frac{2p}{m}$. Therefore, *the point of contact on the tangent having the slope m is* $\left(\frac{p}{m^2}, \frac{2p}{m}\right)$.

EXERCISES

1. Show that the coördinates of the points of contact of the tangents to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, having the slope m, are $\left(\frac{-a^2m}{k}, \frac{-b^2}{k}\right)$, where k has the values $\pm\sqrt{a^2m^2-b^2}$.

2. Show that the coördinates of the points of contact of the tangents to the circle $x^2+y^2=a^2$, having the slope m, are $\left(\frac{-a^2m}{k}, \frac{a^2}{k}\right)$, where k has the values $\pm a\sqrt{m^2+1}$.

3. Find the coördinates of the points of contact of the tangents to the conics in exercise 1, Art. 95.

4. Find the coördinates of the points of contact of the tangents to the pairs of conics in exercise 2, Art. 95.

97. Equation of a tangent in terms of the coördinates of the point of contact.

First method. Let $P(x_1, y_1)$ be the point of contact of a tangent to the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

Then, by the preceding article,

$$x_1=\frac{-a^2m}{k} \text{ and } y_1=\frac{b^2}{k}.$$

Eliminating k by division, we have

$$\frac{y_1}{x_1}=\frac{-b^2}{a^2m}, \text{ or } m=-\frac{b^2x_1}{a^2y_1}.$$

Since the tangent passes through the point of contact, its equation is

$$y-y_1=m(x-x_1)=-\frac{b^2x_1}{a^2y_1}(x-x_1). \quad (1)$$

Clearing of fractions and remembering that $b^2x_1^2 + a^2y_1^2 = a^2b^2$, since the point of contact is on the ellipse, equation (1) reduces to

$$\boldsymbol{\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1.} \tag{2}$$

For the parabola, if $P(x_1, y_1)$ is the point of contact of a tangent, we have seen that $y_1 = \frac{2p}{m}$. Therefore the slope of the tangent is $\frac{2p}{y_1}$, and its equation is

$$y - y_1 = m(x - x_1) = \frac{2p}{y_1}(x - x_1). \tag{3}$$

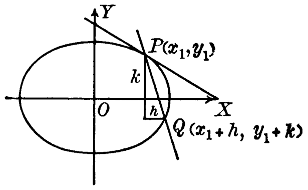

FIG. 71

Clearing of fractions and remembering that $y_1^2 = 4px_1$, we have

$$\boldsymbol{yy_1 = 2p(x + x_1).} \tag{4}$$

Second method. Let a secant meet a curve in the points $P(x_1, y_1)$ and $Q(x_1 + h, y_1 + k)$ (Fig. 71), so that the projections of the segment PQ upon the X- and Y-axes are respectively h and k. The slope of PQ is then $\frac{k}{h}$. The coördinates of P and Q satisfy the equation of the curve. Hence, if the curve is an ellipse as in the figure we have

$$\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} = 1 \quad \text{and} \quad \frac{(x_1 + h)^2}{a^2} + \frac{(y_1 + k)^2}{b^2} = 1. \tag{5}$$

Subtracting the first equation from the second, member from member, we obtain the equation

$$\frac{2hx_1 + h^2}{a^2} + \frac{2ky_1 + k^2}{b^2} = 0,$$

from which we get

$$\frac{k}{h} = -\frac{b^2(2x_1 + h)}{a^2(2y_1 + k)}. \tag{6}$$

As the secant rotates about P (cf. Art. 95), the point Q approaches P along the curve and in the limit coincides with it, and then the secant becomes the tangent at P. But the slope of the secant is constantly equal to the right-hand member of (6). When Q coincides with P, both h and k are zero, and the right-hand member of (6) gives the slope of the tangent at P; that is,

$$m = -\frac{b^2x_1}{a^2y_1}.$$

The equation of the tangent is then found as in the first method.

EXERCISES

1. Write the equation of the tangent to the ellipse $3x^2 + 4y^2 = 19$ at the point (1, 2).

2. Show that the equation of the tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at the point (x_1, y_1) is $\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$.

3. Write the equation of the tangent to the hyperbola $2x^2 - y^2 = 14$ at the point (3, -2).

4. Find the equations of the tangents to the ellipse $16x^2 + 25y^2 = 400$ which pass through the point (3, 4).

5. Write the equation of the tangent to the parabola $y^2 = 6x$ at the point (6, -6).

6. Find the angle which the ellipse $4x^2 + y^2 = 5$ makes with the parabola $y^2 = 8x$ at a point of intersection.

SUGGESTION. Find the equation of the tangent to each curve at a point of intersection and then find the angle which one tangent makes with the other.

7. Show that the equation of the tangent to the circle $x^2 + y^2 = a^2$ at the point (x_1, y_1) is $xx_1 + yy_1 = a^2$.

8. Show that the length of a tangent to the circle $x^2 + y^2 = a^2$, included between the point of contact and the point (x_2, y_2), is $\sqrt{x_2^2 + y_2^2 - a^2}$.

9. Prove that the circles whose equations are $x^2 + y^2 - 8x + 4y + 7 = 0$ and $x^2 + y^2 - 10x - 6y + 21 = 0$ intersect at right angles.

SUGGESTION. Show that the square of the distance between the centers is equal to the sum of the squares of the radii.

10. Using the second method of Art. 97, find the equations of the tangents to the following curves at the points designated:

(*a*) $y^2 = x^3$, at (4, 8).
(*b*) $y = x^2(x - 1)$, at (2, 4).
(*c*) $y^3 = x^2$, at (8, 4).
(*d*) $y^2 = x(x - 1)(x - 2)$, at (3, $\sqrt{6}$).

Draw each curve.

98. Normals. Given any curve, the line drawn perpendicular to a tangent at the point of contact is called the **normal** to the curve at the point of contact.

Let $P(x_1, y_1)$ be the point of contact and m the slope of the tangent at P, then the equation of the normal is

$$y - y_1 = -\frac{1}{m}(x - x_1). \tag{1}$$

For example, the slope of the tangent to the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

at the point (x_1, y_1) is $-\frac{b^2 x_1}{a^2 y_1}$ (Art. 97). Hence the equation of the normal at (x_1, y_1) is

$$y - y_1 = \frac{a^2 y_1}{b^2 x_1}(x - x_1). \tag{2}$$

99. Tangent length, normal length, subtangent, subnormal. Connected with every point on a curve there is a special triangle whose sides are respectively the tangent at the point, the normal at the point, and the X-axis. In Fig. 72, PTN is such a triangle, where PT is the tangent at P, PN is the normal at P, and TN is the X-axis. PT is called the **tangent length** and PN the **normal length.** DP is the ordinate of P, DT is the projection of PT on the X-axis and is called the **subtangent,** DN is the projection of PN on the X-axis and is called the **subnormal.**

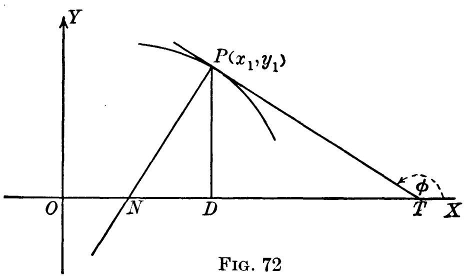

FIG. 72

Let the coördinates of P be x_1, y_1 and the inclination of the tangent be ϕ, then $DP = y_1$ and the angle $DPN =$ angle DTP. Therefore,

$$PT = |y_1 \operatorname{cosec} \phi|,$$
$$PN = |y_1 \sec \phi|,$$

$$DT = |y_1 \cot \phi|,$$
$$DN = |y_1 \tan \phi|,$$

where the bars indicate positive, or absolute, value.

EXERCISES

1. Write the equation of the normal to the circle at the point (x_1, y_1). Note that the normal passes through the center of the circle.

2. Write the equation of the normal to the parabola at point (x_1, y_1). The equation of the normal to the hyperbola at (x_1, y_1).

3. The point $(3, 2)$ lies on the ellipse $x^2 + 4y^2 = 25$. Find the tangent length, the normal length, the subtangent, and the subnormal, at this point. Construct the figure.

4. Find the equation of the normal to the parabola $y^2 = 8x$ which is parallel to the line $2x + 3y = 10$.

5. Prove that the normal to the ellipse or the hyperbola at the point (x_1, y_1) meets the X-axis at a distance e^2x_1 from the center.

6. Show that the subnormal to the parabola $y^2 = 4px$ is constant and equal to $2p$.

7. The line $3x + 8y = 25$ is tangent to the ellipse $x^2 + 4y^2 = 25$. Find the coördinates of the point of contact and write the equation of the normal at this point.

8. The line $mx - 4y = 1$ is tangent to the hyperbola $\frac{x^2}{9} - \frac{y^2}{4} = 1$. Find m and compute the subtangent and subnormal for the point of contact.

100. Reflection properties. The three theorems that follow express what are known as reflection properties.

THEOREM I. *The angle formed by the focal radii drawn to any point of an ellipse is bisected by the normal at that point.*

The equation of the normal at the point (x_1, y_1) is given in (2), Art. 98. From this equation we find the intercept on the X-axis is (Fig. 73)

$$ON = \frac{(a^2 - b^2)x_1}{a^2} = \frac{c^2x_1}{a^2} = \frac{a^2e^2x_1}{a^2} = e^2x_1.$$

But $FO = OF_1 = ae$, and therefore

$$FN = ae + e^2x_1 \quad \text{and} \quad NF_1 = ae - e^2x_1.$$

The ratio of FN to NF_1 is, therefore,

$$\frac{FN}{NF_1} = \frac{a + ex_1}{a - ex_1} = \frac{FP}{F_1P} \text{ (Art. 50).}$$

Hence, we have shown that the normal at P divides the base of the triangle PFF_1 into two segments which are proportional to the adjacent sides. Therefore the normal bisects the angle FPF_1.

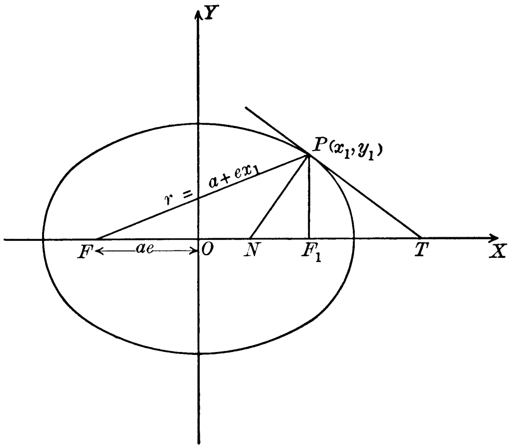

FIG. 73

A ray of light from either focus of an ellipse is reflected from the curve to the other focus.

THEOREM II. *The angle formed by the focal radii drawn to any point of an hyperbola is bisected by the tangent at that point.*

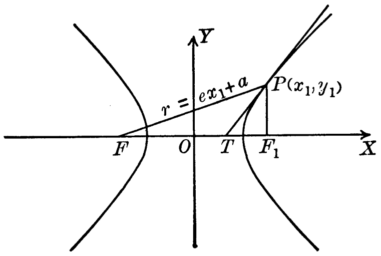

FIG. 74

The equation of the tangent at (x_1, y_1) is

$$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1.$$

Hence the intercept on the X-axis is (Fig. 74),

$$OT = \frac{a^2}{x}.$$

We can now show that

$$\frac{FT}{TF_1} = \frac{PF}{PF_1}.$$

For, $\quad FT = ae + \frac{a^2}{x_1}, \quad TF_1 = ae - \frac{a^2}{x_1};$

and (Art. 52) $\quad PF = ex_1 + a, \quad PF_1 = ex_1 - a.$

Hence the tangent at P divides the base of the triangle PFF_1 into two segments which are proportional to the adjacent sides, and therefore bisects the angle FPF_1.

Theorem III. *Any tangent to the parabola* $y^2 = 4px$ *bisects the angle formed by the focal radius drawn to the point of contact and a line through the point of contact parallel to the X-axis.*

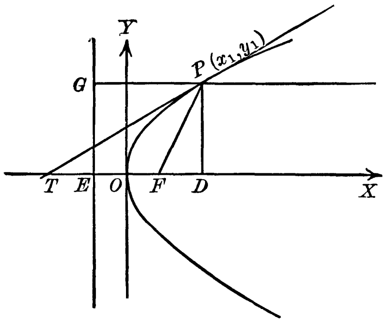

Fig. 75

The equation of the tangent to the parabola at the point $P(x_1, y_1)$ (Fig. 75) is (Art. 97)

$$yy_1 = 2p(x + x_1).$$

Hence, the tangent meets the X-axis at the point $T \equiv (-x_1, 0)$. Therefore $TO = OD$, and, if EG is the directrix and F the focus, $EO = OF$. Hence, $TF = ED$. But $ED = FP$ (definition of the parabola), and, consequently, TFP is an isosceles triangle. The angles PTF and FPT are therefore equal. If PG is parallel to the X-axis, the angles PTF and TPG are equal. Consequently, the tangent PT bisects the angle FPG.

EXERCISES

1. Two parabolas have a common axis and a common focus, and extend in opposite directions. Show that they intersect at right angles.

2. Given the focus and the vertex of a parabola, but not the constructed curve, show how to draw the two tangents through a given point P.

Suggestion. Let F be the focus and A the vertex. Draw the line AF and construct the directrix. With P as center and PF as radius, draw a circle meeting the directrix in D and D_1. The perpendicular bisectors of DF and D_1F are the required tangents. The tangents thus constructed meet the perpendiculars to the directrix at D and D_1 in the points of contact.

3. Show that an ellipse and an hyperbola having the same foci intersect at right angles.

4. Having given the length of the transverse axis and the distance between the foci of an ellipse, or an hyperbola, show how to construct the tangents to the curve from a given point P.

SUGGESTION. Let AB be the given transverse axis. Locate the foci on AB at the points F and F_1. With P as center draw a circle through the nearer focus F_1, and with F as center and AB as radius, a second circle meeting the first in the points D and D_1. The perpendicular bisectors of DF_1 and D_1F_1 are the required tangents. These tangents meet the lines FD and FD_1 in the points of contact.

5. Why is light emanating from the focus of a parabolic mirror reflected in parallel rays? What use is made of this fact?

DIAMETERS

101. Definition. Any line through the center of a circle, an ellipse, or an hyperbola is called a **diameter** of the curve.

Any line perpendicular to the directrix of a parabola is called a **diameter** of the curve. That diameter of the parabola which passes through the focus is the **axis.**

The circle, ellipse, and hyperbola are called **central conics;** the parabola has no center, and is therefore called **noncentral.**

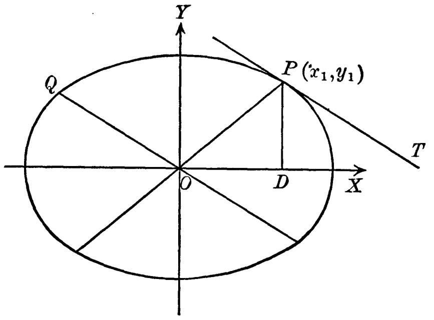

FIG. 76

102. Conjugate diameters. If P (x_1, y_1) is any point on a central conic and PT is the tangent at P, then the diameter through P and the diameter parallel to PT are called **conjugate diameters.** Thus, PO and QO are conjugate diameters (Figs. 76 and 77).

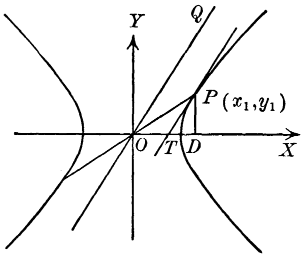

FIG. 77

THEOREM. *If m and m' are the slopes of a pair of conjugate diameters, then*

$$mm' = \mp \frac{b^2}{a^2},$$

according as the conic is an ellipse or an hyperbola.

In either Fig. 76 or Fig. 77, the slope of PO is

$$m = \frac{y_1}{x_1},$$

and the slope of QO, = the slope of PT, $= m' = \mp \frac{b^2 x_1}{a^2 y_1}$,

according as the conic is an ellipse or an hyperbola. Consequently,

$$mm' = \mp \frac{b^2}{a^2}.$$

Since the product of the slopes is independent of the coördinates of P, it follows, in case of the ellipse, that the tangent at Q is parallel to the diameter PO.

EXERCISES

1. Given any diameter of an ellipse or an hyperbola, construct its conjugate diameter.

2. The point $(\frac{3\sqrt{3}}{2}, 1)$ lies on the ellipse $4x^2 + 9y^2 = 36$. Find the equations of the diameter through the point and its conjugate diameter.

3. The point (x_1, y_1) lies on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Show that the equations of the diameter through the point and its conjugate diameter are, respectively,

$$y_1 x - x_1 y = 0 \text{ and } \frac{x_1 x}{a^2} + \frac{y_1 y}{b^2} = 0.$$

4. The point (x_1, y_1) lies on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. Find the equations of the diameter through the point and its conjugate diameter.

5. If a diameter of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ meets the curve in the point (x_1, y_1), show that the conjugate diameter meets the curve in the points $\left(-\frac{ay_1}{b}, \frac{bx_1}{a}\right)$ and $\left(\frac{ay_1}{b}, -\frac{bx_1}{a}\right)$.

6. Prove that the sum of the squares of any two conjugate semidiameters of an ellipse is equal to the sum of the squares of the semiaxes.

7. Show that conjugate diameters of a circle are always at right angles to each other.

8. What is the relation between the slopes of conjugate diameters of the equilateral hyperbola $(b = a)$?

9. Two chords are drawn from any point of an ellipse or an hyperbola to the extremities of a diameter. Show that the diameters bisecting these chords are conjugate diameters.

SUGGESTION. Let the coördinates of the point be x_2 and y_2, and the coördinates of one extremity of the diameter be x_1 and y_1. The coördinates of the other extremity are then $-x_1$ and $-y_1$. Show that the product of the slopes of the diameters bisecting the chords is $-\frac{b^2}{a^2}$, for the ellipse, and $+\frac{b^2}{a^2}$ for the hyperbola.

10. Show that the area of a parallelogram inscribed in the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ whose diagonals are conjugate diameters is $2\,ab$.

SUGGESTION. Let O be the center of the ellipse, $P(x_1, y_1)$ and Q, two adjacent vertices of the parallelogram. The coördinates of Q may then be taken as $\left(\frac{ay_1}{b},\ -\frac{bx_1}{a}\right)$ (exercise 6). The area of the parallelogram is four times the area of the triangle POQ.

11. Prove that the axes of an ellipse or an hyperbola form a pair of conjugate diameters.

SUGGESTION. As the slope of one diameter approaches zero, what does the slope of the conjugate diameter approach ?

12. Can a pair of conjugate diameters of an ellipse or an hyperbola ever be at right angles unless they are the axes ? Why ?

103. The locus of the middle points of a system of parallel chords.

THEOREM. *The locus of the middle points of a system of parallel chords of any conic is a diameter of the conic.*

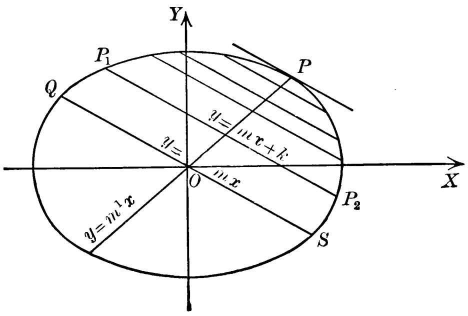

FIG. 78

Let $y = mx + k$ be the equation of any line meeting the conic in the points P_1 and P_2. If the conic is an ellipse, as in Fig. 78, the x-coördinates of the points P_1 and P_2 are the roots of equation (B), Art. 95, and if x_1 and x_2 represent these roots, then the x-coördinate of the middle point of the chord P_1P_2 is given by the equation

$$x' = \frac{x_1 + x_2}{2}.$$

But the sum of the roots of (B) is $-\frac{2\,a^2mk}{a^2m^2 + b^2}$ Hence,

$$x' = -\frac{a^2mk}{a^2m^2 + b^2},$$

and since the middle point is on the chord,

$$y' = mx' + k = \frac{b^2k}{a^2m^2 + b^2}.$$

Therefore, whatever value is given to k, the coördinates of the middle point, x' and y', satisfy the equation

$$y = -\frac{b^2}{a^2m}\,x. \qquad (1)$$

But (1) is the equation of a straight line passing through the center O and is therefore a diameter of the curve. The line PO represents this diameter.

If $k = 0$, the line P_1P_2 assumes the position QS, which is also a diameter of the curve. Since the product of the slopes of PO and QS is $-\frac{b^2}{a^2}$, these diameters are conjugate to each other. Combining this result with the theorem in the preceding article, we conclude that all the chords parallel to PO are bisected by QS.

The theorem is proved for the other conics in a similar way, making use of equations (A), (C), and (D) of Art. 95.

EXERCISES

1. Find the equation of the diameter of the hyperbola $x^2 - 8\,y^2 = 96$ bisecting all the chords parallel to the line $3\,x - 8\,y = 10$.

2. Find the equation of the diameter of the parabola $y^2 = 6\,x$ bisecting all the chords parallel to the line $x + 3\,y = 8$.

3. What is the equation of the chord of the ellipse $9\,x^2 + 36\,y^2 = 324$ which is bisected by the point (4, 2)?

4. Find the equation of the chord of the ellipse $13\,x^2 + 11\,y^2 = 113$ which passes through the point (1, 3) and is bisected by the diameter $2\,y = 3\,x$.

5. What is the equation of the chord of the parabola $y^2 = 6x$ which is bisected by the point (4, 3) ?

6. Find the equation of the diameter of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ which bisects all the chords of slope m.

7. Prove that the diagonals of any circumscribing parallelogram to an ellipse form a pair of conjugate diameters of the curve.

SUGGESTION. Let m and n represent the slopes of a pair of adjacent sides of the parallelogram. Show that the slopes of the diagonals are

$$\frac{mk_1 - nk}{k_1 - k} \text{ and } \frac{mk_1 + nk}{k_1 + k},$$

where $k^2 = a^2m^2 + b^2$ and $k_1{}^2 = a^2n^2 + b^2$. The product of the slopes is therefore $-\frac{b^2}{a^2}$. The diagonals pass through the center of the ellipse and are therefore diameters.

8. Prove exercise 9 of the preceding article by showing that the diameter bisecting one chord is parallel to the other.

POLES AND POLAR LINES

104. Definitions. It has been shown that the equation of a tangent to a conic can be expressed in terms of the coördinates of the point of contact. For example, we saw in Art. 97 that the equation of the tangent to the parabola at the point $P(x_1, y_1)$ is

$$yy_1 = 2p(x + x_1). \tag{1}$$

This equation is the equation of a straight line, whatever values are given to x_1 and y_1, and is the equation of a tangent to the parabola only when the point $P(x_1, y_1)$ is on the curve.

In general (1) is the equation of a straight line called the **polar line** of $P(x_1, y_1)$ with respect to the parabola $y^2 = 4px$. The point $P(x_1, y_1)$ is called the **pole**. If the pole is on the curve, the polar line is tangent to the curve at the pole.

Similarly, the equation of the polar line of any point with respect to any conic can be written at once. For example, the equation of the polar line of $P(x_1, y_1)$ with respect to the ellipse is

$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1. \tag{2}$$

We are led, then, to the following definition:

The polar line of $P(x_1, y_1)$ *with respect to a given conic is that line whose equation has the same form as the equation of the tangent to this conic when* $P(x_1, y_1)$ *is the point of tangency.*

As an example, we will write the equation of the polar line of (1, 3) with respect to the circle $x^2 + y^2 = 4$. Here the equation of the polar line of any point (x_1, y_1) is

$$xx_1 + yy_1 = 4.$$

Hence, the polar line of (1, 3) is

$$x + 3y = 4.$$

The student should draw the figure illustrating this example.

As a second example, find the coördinates of the pole of $5x - 4y + 20 = 0$ with respect to the ellipse

$$\frac{x^2}{25} + \frac{y^2}{16} = 1.$$

Here the polar line of any point (x_1, y_1) is

$$\frac{xx_1}{25} + \frac{yy_1}{16} = 1.$$

If this equation and the given equation are the equations of the same straight line, we must have (Art. 83)

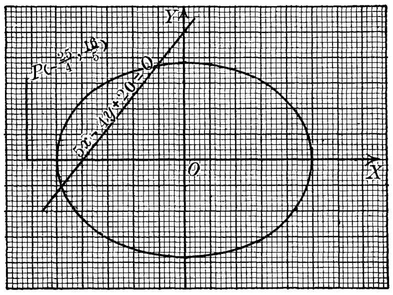

FIG. 79

$$\frac{x_1}{25} = 5k, \quad \frac{y_1}{16} = -4k, \quad \text{and} \quad -1 = 20k,$$

where k is the common ratio. From these equations, we find that $x_1 = -\frac{25}{4}$ and $y_1 = \frac{16}{5}$. Hence the required pole is the point $(-\frac{25}{4}, \frac{16}{5})$ (Fig. 79).

EXERCISES

1. Write the equation of the polar line of each of the following points:

1. $(1, -2)$ with respect to $x^2 + 4y^2 = 16$.
2. $(6, -4)$ with respect to $y^2 = 4x$.
3. $(-2, 2)$ with respect to $5x^2 - 8y^2 = 24$.
4. $(2, -3)$ with respect to $5x^2 + 4y^2 = 10$.

2. Find the coördinates of the pole of the line $3x - 2y = 5$ with respect to the circle $x^2 + y^2 = 25$.

3. What are the cöordinates of the pole of $5x + 4y = 7$ with respect to the ellipse $x^2 + 2y^2 = 10$?

4. Find the coördinates of the pole of the line $x - y = 10$ with respect to the parabola $y^2 = 8x$.

5. What are the coördinates of the pole of the line $Ax + By + C = 0$ with respect to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$?

6. Through the point (x_1, y_1) a line is drawn parallel to the polar line of the point with respect to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. What are the coördinates of the pole of this parallel?

105. Geometric properties of poles and polar lines. A point is **outside** a conic when two tangents can be drawn from the point to the conic. A point is **inside** a conic when no tangents can be drawn from it to the conic.

THEOREM I. *If the point (x_1, y_1) is outside a conic, its polar line with respect to the conic passes through the points of contact of the tangents drawn from the point.*

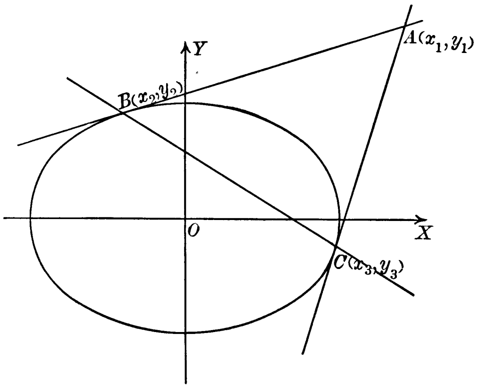

FIG. 80

Let the conic be the ellipse; the proof is similar for the other conics. Let $B(x_2, y_2)$ and $C(x_3, y_3)$ be the points of contact of tangents drawn from $A(x_1, y_1)$ (Fig. 80). The equations of the tangents at B and C are, respectively,

$$\frac{xx_2}{a^2} + \frac{yy_2}{b^2} = 1 \text{ and } \frac{xx_3}{a^2} + \frac{yy_3}{b^2} = 1.$$

Since the tangents pass through A, the coördinates of A satisfy each of the above equations. Hence,

$$\frac{x_1x_2}{a^2} + \frac{y_1y_2}{b^2} = 1 \text{ and } \frac{x_1x_3}{a^2} + \frac{y_1y_3}{b^2} = 1.$$

But these same equations result from substituting the coördinates of the points B and C in the equation of the polar line of A with respect to the ellipse. Consequently the polar line of A passes through B and C.

Exercise. Prove Theorem I for each of the other three conics.

THEOREM II. *If P and Q are two points in the plane such that the polar line of P with respect to a given conic passes through Q, then the polar line of Q passes through P.*

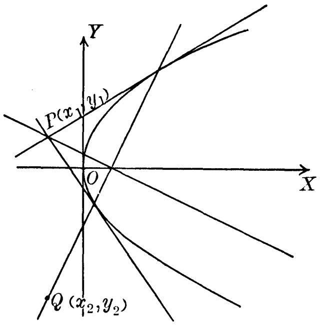

FIG. 81

Take the parabola $y^2 = 4\,px$ (Fig. 81) for the given conic. A similar proof establishes the theorem for the other conics. Let the coördinates of P be x_1, y_1 and the coördinates of Q, x_2, y_2. The equation of the polar line of P is, then,

$$yy_1 = 2\,p(x + x_1).$$

Since this line passes through Q, we have

$$y_2y_1 = 2\,p(x_2 + x_1).$$

But this same equation results from substituting the coördinates of P in the equation of the polar line of Q. Hence, the polar line of Q passes through P.

Exercise. Prove Theorem II for each of the other three conics.

THEOREM III. *If a line through the pole A meets the polar line in C and the conic in B and D, then*

$$\frac{AB}{BC} = -\frac{AD}{DC}.$$

Let the coördinates of A (Fig. 82) be x_1, y_1 and the coördinates of C be x_2, y_2. The coördinates of the point B, dividing the segment AC in the ratio $AB : BC = r$, are

$$x = \frac{x_1 + rx_2}{1 + r} \text{ and } y = \frac{y_1 + ry_2}{1 + r} \text{ (Art. 17).}$$

But B lies on the conic. Hence, if the conic is an ellipse as in the figure,

$$\frac{(x_1 + rx_2)^2}{a^2(1 + r)^2} + \frac{(y_1 + ry_2)^2}{b^2(1 + r)^2} = 1.$$

Expanding and arranging according to the powers of r, we have

$$\left(\frac{x_2^2}{a^2} + \frac{y_2^2}{b^2} - 1\right)r^2 + 2\left(\frac{x_1x_2}{a^2} + \frac{y_1y_2}{b^2} - 1\right)r + \left(\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1\right) = 0.$$

We should have been led to the same equation had we taken $AD : DC = r$. Hence the roots of this equation are the ratios in which the curve points divide the segment AC. But, since C is on the polar line of A, the coefficient of r vanishes. Hence the roots are equal but opposite in sign; that is,

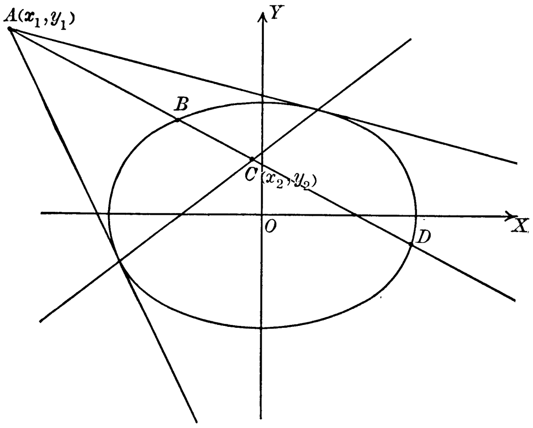

FIG. 82

$$\frac{AB}{BC} = -\frac{AD}{DC}.$$

Four points A, B, C, and D situated on a straight line and such that $\frac{AB}{BC} = -\frac{AD}{DC}$ constitute a **harmonic range**. The segment AC is said to be divided harmonically by B and D.

Exercise. Prove Theorem III for the parabola and for the circle.

THEOREM IV. *The polar line of a focus of a conic is the corresponding directrix.*

We shall establish the theorem for the case of an ellipse, leaving the remaining cases as an exercise for the student. The coördinates of the right-hand focus of an ellipse are $x = ae$ and $y = 0$. Substituting these for x_1 and y_1 in the general equation of the polar line with respect to the ellipse, we have, as the polar line of the focus,

$$x = \frac{a}{e}.$$

But this is the equation of the right-hand directrix.

Exercise. Prove that the directrix of a parabola is the polar line of the focus; that either directrix of an hyperbola is the polar line of the corresponding focus.

THEOREM V. *The line joining a focus of a conic to the intersection of any two tangents, bisects one of the angles formed by the focal radii drawn to the points of contact of the tangents.*

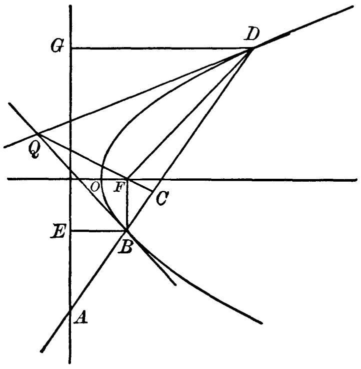

FIG. 83

In Fig. 83, let B and D be the points of contact of tangents from Q, F being a focus. Draw the line BD and let it meet the directrix corresponding to F in A, and the line QF in C. Draw the lines BE and DG perpendicular to the directrix. Since F is the pole of the directrix (Theorem IV) and Q is the pole of the line BD (Theorem I), it follows that QF is the polar line of A (Theorem II). Therefore we have the following equations:

$$\frac{AB}{BC}=\frac{AD}{CD} \text{ (Theorem III, } CD=-DC\text{)},$$

$$\frac{EB}{AB}=\frac{GD}{AD} \text{ (Similar triangles)},$$

$$\frac{BF}{EB}=\frac{DF}{GD} \text{ (Eccentricity, property } A\text{, Art. 94)}.$$

Equating the product of the right-hand members of these equations to the product of their left-hand members, we obtain

$$\frac{BF}{BC}=\frac{DF}{CD}.$$

Hence the point C divides the side BD of the triangle BFD in the ratio $BF:DF$, and consequently the line QF bisects the angle BFD.

EXERCISES

1. Show how Theorem II can be used to construct the pole of any line with respect to a given conic.

SUGGESTION. Construct the polar line of any two points on the given line. Where do these intersect?

2. Two lines are drawn through a point P. The poles of these lines with respect to any conic are the points R and Q. Show that RQ is the polar line of P with respect to the same conic.

3. Given any two lines in the plane such that the first passes through the pole of the second with respect to any conic. Show that the second passes through the pole of the first.

4. Prove that the intersection of any two tangents to an ellipse or an hyperbola is equidistant from the four focal radii that can be drawn to the points of contact.

5. Show that the intersection of any two tangents to a parabola is equidistant from the focal radii to the points of contact and the diameters through the points of contact.

6. In Fig. 75, Art. 100, let PO meet the directrix in M, and PT meet the vertical tangent in Q. Show that QF is the polar line of M.

SYSTEMS OF CONICS

106. The asymptotes of the hyperbola. It has already been noticed that the hyperbola has asymptotes (Art. 67). This is a characteristic property of the hyperbola. No other conic has asymptotes.

Solving the standard form of the equation of an hyperbola for y, we have

$$y = \pm \frac{b}{a}\sqrt{x^2 - a^2}.$$

Hence, as x increases indefinitely, y approaches nearer and nearer to the values $\pm \frac{bx}{a}\cdot$ Therefore

$$\boldsymbol{y = \pm \frac{bx}{a}} \qquad (1)$$

are the equations of the asymptotes.

The equations of the asymptotes can be written

$$\boldsymbol{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0,} \qquad (2)$$

since the coördinates of any point on either asymptote will satisfy equation (2) (cf. Art. 85).

The equation of a tangent to the hyperbola in terms of the slope is (Art. 95, Eq. 7)

$$y = mx + \sqrt{a^2m^2 - b^2}. \tag{3}$$

If, in this equation, m is taken as the slope of either asymptote; namely, $\pm\frac{b}{a}$, the equation becomes $y = \pm\frac{bx}{a}$. For this reason, the asymptotes are often spoken of as *tangents to the hyperbola, the points of contact being infinitely distant.*

Since an asymptote passes through the center, it is a diameter of the hyperbola (Art. 101). The product of the slopes of a pair of conjugate diameters of the hyperbola is $\frac{b^2}{a^2}$ (Art. 102), and therefore each asymptote is its own conjugate diameter, or in other words, an asymptote is a **self-conjugate diameter** of the hyperbola.

107. Conjugate hyperbolas. The two hyperbolas

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, \text{ and } \frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$$

are called **conjugate hyperbolas.** The transverse and conjugate axis of the one are respectively the conjugate and transverse axis of the other. Either hyperbola is conjugate to the other, but it

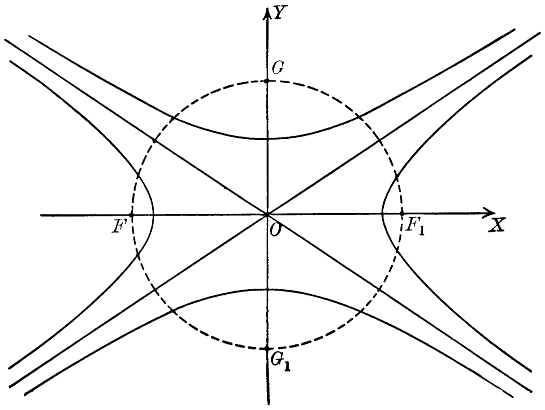

FIG. 84

is convenient to speak of the first as the **primary**, and the second as the **conjugate**, hyperbola.

The foci of the conjugate hyperbola are on the Y-axis, and, since $c = \sqrt{a^2 + b^2}$ is the same for each hyperbola, the four foci lie on a circle of radius c and center at the origin (Fig. 84).

The eccentricity of the conjugate hyperbola differs from the eccentricity of the primary hyperbola. The former is $\frac{c}{b}$ and the latter is $\frac{c}{a}\cdot$

The asymptotes of the conjugate hyperbola coincide with the asymptotes of the primary hyperbola. For, from the equation of the conjugate hyperbola, we have

$$y = \pm \frac{b}{a}\sqrt{x^2 + a^2}.$$

Hence, as x increases indefinitely, the curve approaches nearer and nearer to the lines $y = \pm \frac{bx}{a}\cdot$ But these are the asymptotes of the primary hyperbola.

EXERCISES

1. Show that the foot of the perpendicular from a focus of an hyperbola on either asymptote is at a distance a from the center and b from the focus.

2. Show that the circle of radius, b, whose center is at a focus of an hyperbola, is tangent to the asymptotes at the points where they cut the corresponding directrix.

3. Show that the product of the perpendiculars let fall from any point of an hyperbola on the asymptotes is constant.

4. Write the equation of the hyperbola conjugate to $9x^2 - y^2 = 9$, and find the lengths of its semiaxes, its eccentricity, the coördinates of its foci, and the equations of its directrices.

5. If e and e_1 are the eccentricities of two conjugate hyperbolas, show that $\frac{1}{e^2} + \frac{1}{e_1{}^2} = 1$. Also $ae = be_1$.

6. What is the eccentricity of the equilateral hyperbola? Of the conjugate to the equilateral hyperbola?

108. The system of concentric hyperbolas. The equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = k, \tag{1}$$

where k is a variable parameter, is the equation of a **system of concentric hyperbolas** (Fig. 85). All the hyperbolas contained in the system have the same asymptotes, as can be shown by solving (1) for y and then allowing x to increase indefinitely. If k is a negative number, (1) is the equation of a conjugate hyperbola. As k increases in value, the hyperbola approaches the asymptotes closer and closer, and coincides with them when k is zero (Art. 106). When k is an increasing positive number, the hyperbolas are all primary and recede farther and farther from the asymptotes.

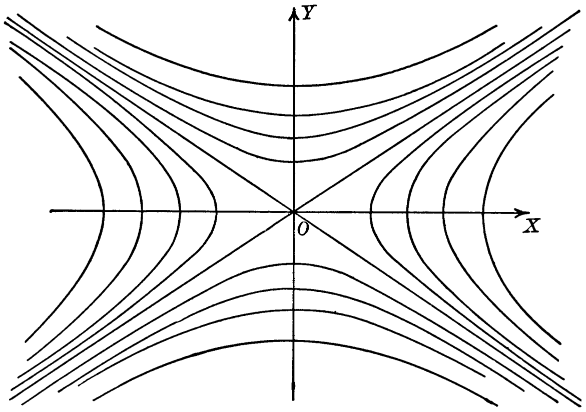

FIG. 85

The contour lines about two contiguous mountain peaks form a rough approximation to such a system of hyperbolas.

109. The system of confocal conics. Conics having the same foci are called **confocal**. The equation

$$\frac{x^2}{a^2-k}+\frac{y^2}{b^2-k}=1, \tag{1}$$

where k is a variable parameter, is the equation of a **system of confocal conics** (Fig. 86). For, suppose $a > b$, then for every value of $k < b^2$, (1) is the equation of an ellipse. The distance from center to focus is the same for all these ellipses, since

$$c=\sqrt{(a^2-k)-(b^2-k)}=\sqrt{a^2-b^2}.$$

As k approaches b^2, the semiconjugate axis of the ellipse approaches zero and the semitransverse axis approaches $\sqrt{a^2-b^2}$,

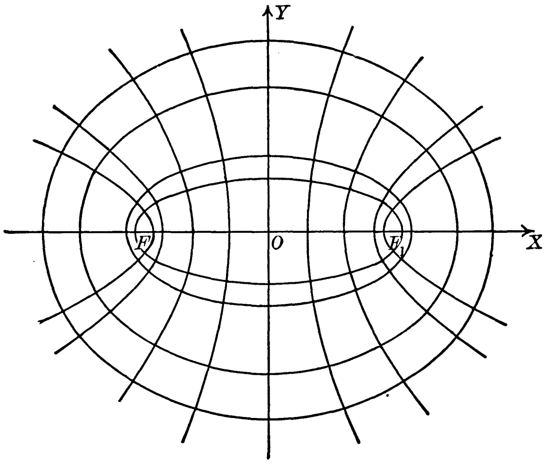

FIG. 86

and therefore the ellipse shrinks to the segment of the X-axis contained between the foci.

If $b^2 < k < a^2$, (1) is the equation of an hyperbola whose semiaxes are $\sqrt{a^2-k}$ and $\sqrt{k-b^2}$. The distance from center to focus is, in this case,

$$c=\sqrt{(a^2-k)+(k-b^2)}=\sqrt{a^2-b^2},$$

and therefore these hyperbolas are confocal with the ellipses.

For any value of $k > a^2$, equation (1) is satisfied for no real values of x and y. In this case, (1) is said to be the equation of an **imaginary ellipse**.

Each hyperbola of the system cuts every ellipse at right angles and vice versa (Exercises, Art. 100).

If two sets of curves are so related that each curve of either set intersects all the curves of the other set at right angles, the two sets of curves are said to be *orthogonal* to each other. Thus the hyperbolas and ellipses of the system of confocal conics form two sets of curves orthogonal to each other. Sets of orthogonal curves are of great importance in mathematical physics, since they represent fields of force.

EXERCISES

1. What are the equations of the two conics of the system

$$\frac{x^2}{(9-k)} + \frac{y^2}{(4-k)} = 1$$

which pass through the point $\left(\frac{4\sqrt{15}}{5}, \frac{\sqrt{5}}{5}\right)$?

2. Show that the equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = k$ is the equation of a system of concentric ellipses, k being a variable parameter. Discuss the equation for various values of k.

3. Show that the equation $y^2 = 4k(x + k)$ is the equation of a system of confocal parabolas. Discuss the equation for various values of k.

4. What system of conics is given by the equation $\frac{x^2}{a^2} + \frac{y^2}{k^2} = 1$, k being a variable parameter? Show that the x-intercept of a tangent to any one of these conics is independent of k. How can this fact be utilized to construct the tangent at any point on one of the conics of the system?

5. Discuss the system of circles given by the equation $x^2 + y^2 - a^2 - 2ky = 0$, k being a variable parameter.

6. Discuss the system of circles given by the equation $x^2 + y^2 + a^2 - 2mx = 0$, m being a variable parameter.

7. Show that the two systems of circles in exercises 5 and 6 form two sets of circles orthogonal to each other. Draw a figure illustrating this exercise.

MISCELLANEOUS EXERCISES

1. Find the equations of the tangents to the ellipse $x^2 + 4y^2 = 16$ which pass through the point (2, 3).

2. Find the equations of the tangents to the hyperbola $2x^2 - 3y^2 = 18$ which pass through the point $(4, -\sqrt{5})$.

3. Find the coördinates of the points of contact of the tangents in exercises 1 and 2.

4. For what value of k is $y = 2x + k$ a tangent to the hyperbola $x^2 - 4y^2 = 4$?

5. For what value of m is $y = mx + 2$ a tangent to the ellipse $x^2 + 4y^2 = 1$?

6. What relation connects A, B, and C, if $Ax + By + C = 0$ is a tangent to the parabola $y^2 = 4x$?

7. Are the following points on, inside, or outside the hyperbola $4x^2 - y^2 = 4$? (*a*) $(\frac{1}{2}, 3)$, (*b*) $(2, 1)$, (*c*) $(3.25, 3)$.

8. The coördinates of one extremity of a diameter of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ are $x_1 = a \cos \theta_1$ and $y_1 = b \sin \theta_1$. Show that the coördinates of one extremity of the conjugate diameter are given by the equations $x_2 = -a \sin \theta_1$ and $y_2 = b \cos \theta_1$.

9. Show that the segments of any line contained between an hyperbola and its asymptotes are equal in length.

10. Find the equation of the tangent to the parabola $y^2 = 4px$ which has equal intercepts.

11. The earth's orbit is an ellipse whose eccentricity is .01677 and whose major semiaxis is 93 million miles, the sun being at one focus. Find the greatest and the least distance from the earth to the sun.

12. Find the angle which one diameter of an ellipse makes with its conjugate diameter.

13. A comet moves in a parabolic orbit with the sun at the focus. If the comet is 2 million miles from the sun when the line from sun to comet makes an angle of 60° with the axis of the orbit, find the least distance from sun to comet.

14. Show that the bisector of the angle formed by lines joining any point of an equilateral hyperbola to the vertices is parallel to an asymptote.

15. Find the equation of the locus of the mid-points of chords drawn from one end of the major axis of an ellipse.

16. Show that the ordinate of the intersection of any two tangents to the parabola $y^2 = 4px$ is the arithmetic mean of the ordinates of the points of contact, and the abscissa is the geometric mean of the abscissas of the points of contact.

17. Find the equation of the locus of the intersection of two tangents to the parabola $y^2 = 4px$, if the sum of the slopes of the tangents is constant.

18. Show that the angle formed by any two tangents to the parabola is half the angle formed by the focal radii to the points of contact.

19. Any two perpendicular lines are drawn from the vertex of a parabola. Show that the line joining their other points of intersection with the parabola cuts the axis at a fixed point.

20. Show that the tangents to the parabola at the extremities of any chord intersect on the diameter bisecting the chord.

21. Show that the eccentricity of an hyperbola is equal to the secant of half the angle between the asymptotes.

22. Show that the tangents at the vertices of an hyperbola intersect the asymptotes at points on the circle about the center and passing through the foci.

23. Show that the product of the distances from the center of an hyperbola to the intersections of any tangent with the asymptotes is constant.

CHAPTER VIII

LOCI OF THE SECOND ORDER EQUATIONS NOT IN STANDARD FORM

110. Translation of the coördinate axes. If the coördinate axes are translated by means of equations (1), Art. 77, and then the primes are dropped, the standard forms of the equations of the several conics become:

Circle: $$(x+h)^2+(y+k)^2=r^2, \quad (1)$$

Ellipse: $$\frac{(x+h)^2}{a^2}+\frac{(y+k)^2}{b^2}=1, \quad (2)$$

Primary hyperbola: $$\frac{(x+h)^2}{a^2}-\frac{(y+k)^2}{b^2}=1, \quad (3)$$

Conjugate hyperbola: $$\frac{(x+h)^2}{a^2}-\frac{(y+k)^2}{b^2}=-1, \quad (4)$$

Parabola, X-axis parallel to the axis of the curve:

$$(y+k)^2=4p(x+h), \quad (5)$$

Parabola, Y-axis parallel to the axis of the curve:

$$(x+h)^2=4p(y+k), \quad (6)$$

On the other hand, if an equation of the second degree can be reduced to one or the other of the above forms, the locus is the corresponding conic. The center of the circle, the ellipse, or the hyperbola is then the point $(-h, -k)$, and the vertex of the parabola is the point $(-h, -k)$, Figs. 87 and 88.

As an example, take the equation

$$9x^2+4y^2+54x-16y+61=0.$$

Completing the squares of the terms in x and the terms in y separately, the equation can be written

$$9(x+3)^2+4(y-2)^2=36.$$

Comparing with (2), we see that the locus is an ellipse whose center is the point $(-3, 2)$ and whose semiaxes are 2 and 3.

If any one of the equations (1) to (6) is expanded and cleared of fractions, it is seen to be a special case of the equation

$$\boldsymbol{ax^2 + by^2 + 2\,gx + 2\,fy + c = 0,} \tag{7}$$

where a, b, g, f, and c depend upon h and k. We are led, then, to the following

THEOREM. *The equation of a conic referred to coördinate axes parallel to the axes of the curve (the axis and tangent at vertex in case of the parabola) has the form* (7).

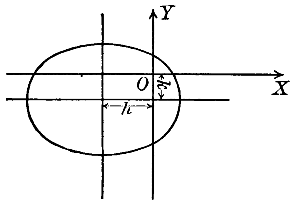

FIG. 87

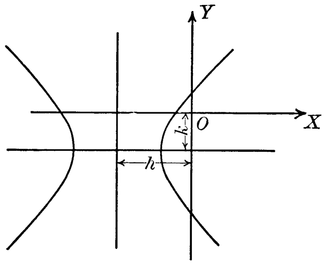

FIG. 88

111. Discussion of the equation $\boldsymbol{ax^2 + by^2 + 2\,gx + 2\,fy + c = 0}$.

The question now arises: Is

$$ax^2 + by^2 + 2\,gx + 2\,fy + c = 0 \tag{1}$$

the equation of a conic for any given set of values of the coefficients? We shall answer this question by a discussion of the equation (cf. Art. 46).

General case. Let us first suppose that none of the coefficients is equal to zero, and we may further suppose without loss of generality, that a is a positive number. For, if a is a negative number in any particular case, we can change the signs of all the terms in the equation. Equation (1) can then be written in the form

$$a\left(x + \frac{g}{a}\right)^2 + b\left(y + \frac{f}{b}\right)^2 = \frac{g^2}{a} + \frac{f^2}{b} - c. \tag{2}$$

Denote the right-hand member of (2) by D, then the nature of the locus will depend upon the signs of b and D. Thus, if b is negative, the locus is an hyperbola which is primary or conjugate according as D is positive or negative (cf. Art. 107). If b and D are positive, the locus is an ellipse. There is no locus if b is positive and D is negative, for the sum of two positive numbers can never be negative. The locus is then said to be *imaginary*.

If either a or b is zero, the above method fails. But if a is zero, while b and g are different from zero, (1) can be written

$$\left(y+\frac{f}{b}\right)^2=-\frac{2g}{b}\left(x+\frac{bc-f^2}{2bg}\right); \tag{3}$$

and if b is zero, while a and f are different from zero, (1) can be written

$$\left(x+\frac{g}{a}\right)^2=-\frac{2f}{a}\left(y+\frac{ac-g^2}{2af}\right). \tag{4}$$

In either case, the locus is a parabola, as we see on comparison with equations (5) and (6) of the preceding article.

Special cases. If D is zero and b is negative, the left-hand member of (2) is the product of two linear expressions, and therefore the locus of (2), and consequently the locus of (1), is a pair of intersecting straight lines (cf. Art. 85). If b is positive, the locus consists of the single point, $\left(-\frac{g}{a}, -\frac{f}{b}\right)$, since this is the only point whose coördinates will then satisfy (2).

Finally, if a and g, or b and f, are each equal to zero, equation (1) contains but one of the variables. For example, if b and f are each equal to zero, (1) becomes

$$ax^2+2gx+c=0. \tag{5}$$

If the roots of (5) are real and distinct, the locus consists of a pair of lines parallel to the Y-axis. If the roots of (5) are equal, the locus consists of a single line parallel to the Y-axis. If the roots of (5) are imaginary, there is no locus, but we shall say, in this case, that the locus consists of a pair of imaginary lines. In these special cases, the locus is said to be **degenerate.**

We shall find it convenient to say that the locus of (1) *is a conic, but that in certain cases, the conic is imaginary, or degenerates into a single line, or into a pair of real or imaginary lines, or consists of a single point.*

As an example of the foregoing analysis, consider the equation

$$9x^2 - 16y^2 - 36x + 96y - 108 = 0.$$

Here we find that a is positive, D is zero, and b is negative. Therefore the locus consists of two intersecting lines. The equations of these lines are

$$3(x - 2) \pm 4(y - 3) = 0,$$

and they intersect in the point (2, 3).

As a second example, consider the equation

$$9x^2 + 2y^2 - 18x + 8y + 17 = 0.$$

In this case, a is positive, D is zero, and b is positive. Hence the locus consists of a single point. Completing the squares of the terms in x and the terms in y separately, the equation becomes

$$9(x - 1)^2 + 2(y + 2)^2 = 0.$$

Therefore the point $(1, -2)$ is the only point whose coördinates satisfy the given equation.

A summary of the possible loci of the equation

$$ax^2 + by^2 + 2gx + 2fy + c = 0$$

is given in the following table, where $D = \frac{g^2}{a} + \frac{f^2}{b} - c.$

$a \neq 0,\ b \neq 0$		$a = 0$ (or $b = 0$)	
$a > 0$ $b > 0$	$a > 0$ $b < 0$	$a = 0,\ g \neq 0,\ b \neq 0$ (or $b = 0,\ f \neq 0,\ a \neq 0$)	$a = g = 0,\ b \neq 0$ (or $b = f = 0,\ a \neq 0$)
$D > 0$ Ellipse	$D > 0$ Hyperbola (primary)	Parabolas	A pair of real parallel lines, a single line, or a pair of imaginary lines; according as the roots of $by^2 + 2fy + c = 0$ (or $ax^2 + 2gx + c = 0$) are real and distinct, equal, or imaginary.
$D = 0$ Point	$D = 0$ Intersecting lines		
$D \quad 0$ Imaginary	$D < 0$ Hyperbola (conjugate)		

In the following exercises, use is to be made of this table.

EXERCISES

1. Determine the nature of the loci of the following equations. Find the coördinates of the center and the coördinates of the foci of each ellipse or hyperbola; the coördinates of the vertex and the coördinates of the focus of each parabola. Make a sketch of each curve.

(*a*) $2x^2 + 3y^2 - 6x + 4y = 10.$ (*b*) $x^2 + 2y^2 - 6x + y = 10.$
(*c*) $4x^2 - 3y^2 - 4x + 8 = 0.$ (*d*) $x^2 + 4x - 2y = 15.$
(*e*) $3x^2 - y^2 + 6y = 0.$ (*f*) $y^2 + 2x - 4y = 7.$

2. Determine the nature of the loci of the following equations:

(*a*) $x^2 + y^2 - 4x - 6y + 13 = 0.$ (*b*) $x^2 - y^2 - 4x + 6y + 5 = 0.$
(*c*) $x^2 - 5x + 6 = 0.$ (*d*) $y^2 - 6y + 9 = 0.$
(*e*) $y^2 - 6y + 10 = 0.$ (*f*) $y^2 - 6y + 8 = 0.$

112. The general equation of second degree. The equation

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \tag{1}$$

is called the **general equation of second degree**, because it contains every term that can appear in an equation of the second degree. We will now prove the following theorem.

THEOREM. *The term in xy can be removed from the general equation of second degree by rotating the axes through a positive angle θ, less than* 90°.

Replacing x and y in (1) by their values in terms of x' and y'; namely,

$$x = x'\cos\theta - y'\sin\theta,$$
$$y = x'\sin\theta + y'\cos\theta,$$

(Art. 78), we obtain

$$a'x'^2 + 2h'x'y' + b'y'^2 + 2g'x' + 2f'y' + c = 0, \tag{2}$$

where

$$a' = a\cos^2\theta + 2h\sin\theta\cos\theta + b\sin^2\theta, \tag{3}$$

$$b' = a\sin^2\theta - 2h\sin\theta\cos\theta + b\cos^2\theta, \tag{4}$$

$$\begin{aligned} 2h' &= 2h(\cos^2\theta - \sin^2\theta) - 2(a-b)\sin\theta\cos\theta \\ &= 2h\cos 2\theta - (a-b)\sin 2\theta, \end{aligned} \tag{5}$$

$$2g' = 2g\cos\theta + 2f\sin\theta, \tag{6}$$

$$2f' = 2f\cos\theta - 2g\sin\theta. \tag{7}$$

If, now, we can choose θ so that h' shall equal zero, the term in $x'y'$ will drop out of (2) and the general equation will be reduced to the form

$$a'x'^2 + b'y'^2 + 2g'x' + 2f'y' + c = 0. \tag{8}$$

Putting h' equal to zero in (5), we have for the determination of θ,

$$2h \cos 2\theta - (a-b) \sin 2\theta = 0,$$

from which

$$\tan 2\theta = \frac{2h}{a-b}. \tag{9}$$

Since the tangent of an angle assumes all possible positive and negative values as the angle increases from 0° to 180°, it follows that *it is always possible to find an angle* θ, *less than* 90°, *which will satisfy equation* (9). If the axes are rotated through this angle, the term in $x'y'$ drops out and the general equation is thus reduced to the form (8).

Equation (8) has the same form as that discussed in the preceding article. Therefore we can say that *the locus of the general equation of second degree is a conic, but that this conic may be imaginary, or may consist of a single line, or of a pair of real or imaginary lines, or of a single point.*

The values of the coefficients a' and b' can be found easily from equations (3), (4), and (5). Thus, adding (3) and (4), we have

$$a' + b' = a + b, \tag{10}$$

and subtracting (4) from (3) gives

$$a' - b' = 2h \sin 2\theta + (a-b) \cos 2\theta. \tag{11}$$

Squaring (5) and (11) and then adding, we obtain

$$4h'^2 + (a'-b')^2 = 4h^2 + (a-b)^2. \tag{12}$$

Subtracting (12) from the square of (10) gives

$$a'b' - h'^2 = ab - h^2. \tag{13}$$

When the coördinate axes have been rotated through the angle given by (9), we have seen that h' becomes zero. Hence equations (10) and (13) give respectively the sum and product of the required coefficients. These coefficients are then the roots of the quadratic equation

$$\lambda^2 - (a+b)\lambda + ab - h^2 = 0. \tag{14}$$

The roots of this equation are always real, since the discriminant, $(a+b)^2 - 4(ab-h^2) = (a-b)^2 + 4h^2$, is always positive.

In order to decide which of the roots of (14) to take for a', eliminate $\cos 2\theta$ between (9) and (11), thus obtaining

$$2h(a'-b') = [4h^2 + (a-b)^2]\sin 2\theta. \qquad (15)$$

For $\theta < 90°$, $\sin 2\theta$ is positive. Therefore a' must be so chosen that $a'-b'$ will have the same sign as h.

The roots of (14) will be both different from zero if $ab - h^2 \neq 0$, and will be alike in sign, or unlike in sign, according as their product $ab - h^2$ is positive, or negative.

The values of the coefficients g' and f' can be computed from equations (6) and (7), but the computation is often tedious, and can be avoided frequently by a translation of the axes (Art. 113). If a given equation of the second degree contains no terms of the first degree; that is, if g and f are each equal to zero, then, by (6) and (7), g' and f' are also each equal to zero and the foregoing analysis serves to determine the nature and the position of the locus.

For example, consider the equation

$$x^2 + 2xy + 2y^2 - 4 = 0.$$

Here $\tan 2\theta = \dfrac{2h}{a-b} = \dfrac{2}{1-2} = -2$, from which we get $\theta = 58° 17'$, nearly. If the axes are rotated through the angle θ, the term in $x'y'$ will drop out. The coefficients of x'^2 and y'^2 are the roots of (14) which becomes, in this case,

$$\lambda^2 - 3\lambda + 1 = 0.$$

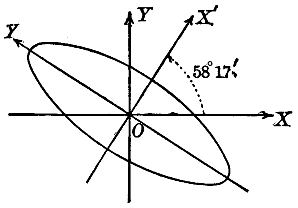

FIG. 89

The roots are $\dfrac{3+\sqrt{5}}{2}$ and $\dfrac{3-\sqrt{5}}{2}$. Since h is positive, $a'-b'$ must be positive, and therefore we choose

$$a' = \frac{3+\sqrt{5}}{2} \text{ and } b' = \frac{3-\sqrt{5}}{2}.$$

The given equation, thus, reduces to

$$\frac{3+\sqrt{5}}{2}x'^2 + \frac{3-\sqrt{5}}{2}y'^2 = 4, \text{ or } \frac{x'^2}{2(3-\sqrt{5})} + \frac{y'^2}{2(3+\sqrt{5})} = 1.$$

The locus is, therefore, an ellipse whose semiaxes are $\sqrt{2(3-\sqrt{5})}$ and $\sqrt{2(3+\sqrt{5})}$. The major axis coincides with the new Y-axis (Fig. 89).

EXERCISES

1. Determine the nature of the locus of the equation $5x^2+2xy+5y^2=12$. Find the angle through which the coördinate axes must be rotated in order to remove the term in xy. Plot the curve and both sets of axes. Find the eccentricity of the curve.

2. What is the locus of each of the following equations?

(*a*) $3x^2-2xy+y^2-6=0$.
(*b*) $3x^2-2xy+y^2=-6$.
(*c*) $3x^2-2xy+y^2=0$.
(*d*) $9x^2-20xy+11y^2-50=0$.
(*e*) $25x^2-60xy+36y^2-81=0$.

3. Reduce the following equations to standard form. Draw the figure for each exercise.

(*a*) $x^2+xy+y^2-1=0$.
(*b*) $x^2+3xy-3y^2-4=0$.
(*c*) $2x^2-12xy-3y^2+14=0$.
(*d*) $43x^2+30xy+59y^2-68=0$.
(*e*) $8x^2-12xy+3y^2-9=0$.

4. The locus of the equation $4x^2+4xy+y^2+k=0$ is two straight lines for any value of k. Discuss the change in these lines as k varies.

5. Show that $3x^2+2hxy+12y^2=3$ is the equation of a system of concentric conics, h being a variable parameter. Discuss the change in the locus as h varies from a great negative number to a great positive number.

113. Removal of the terms of the first degree. If the terms of the first degree can be removed from the general equation of second degree, this can be done by translating the axes, as in Art. 79. Let m and n be the coördinates of the new origin. The equations for translating the axes are then

$$x=x'+m \text{ and } y=y'+n \quad \text{(Art. 77)}.$$

Substituting in the general equation, (1), Art. 112, and arranging according to the powers of x' and y', the resulting equation can be written

$$ax'^2+2hx'y'+by'^2+2\left\{\begin{array}{l} \quad(am+hn+g)x' \\ +(hm+bn+f)y' \end{array}\right\} + \left\{\begin{array}{l} \quad m(am+hn+g) \\ +n(hm+bn+f) \\ +(gm+fn+c) \end{array}\right\}=0. \tag{1}$$

If the terms of first degree can be removed, m and n must satisfy simultaneously the two equations

$$\begin{aligned} \boldsymbol{am+hn+g=0,} \\ \boldsymbol{hm+bn+f=0.} \end{aligned} \tag{2}$$

The values of m and n derived from these equations are,

$$\begin{aligned} m &= \frac{hf - bg}{ab - h^2}, \\ n &= \frac{hg - af}{ab - h^2}. \end{aligned} \tag{3}$$

We now have three cases to consider (cf. Art. 83):

1. Equations (2) are consistent and have but one common solution if, and only if, $ab - h^2$ is not equal to zero. For then equations (3) give but a single pair of values for m and n.

2. Equations (2) are inconsistent, and therefore have no common solution, if $ab - h^2 = 0$ and $hf - bg \neq 0$. For then $hg - af \neq 0$ and the numerators in (3) do not vanish while the denominator is equal to zero.

3. Equations (2) are dependent and therefore have an indefinite number of common solutions if $ab - h^2 = 0$ and $hf - bg = 0$. For then $hg - af = 0$ and both the numerator and the denominator of each fraction in (3) vanishes and any pair of values of m and n that satisfies one of the equations (2) also satisfies the other.

We shall consider the three cases separately, and we assume that in no case is h equal to zero. For, if h is equal to zero, the general equation has the form discussed in Art. 111.

114. First case, $ab - h^2 \neq 0$. Central conics. In this case the terms of first degree can be removed. Setting the values of m and n from (3), Art. 113, in (1) and dropping the primes, (1) becomes

$$ax^2 + 2hxy + by^2 + \frac{g(hf - bg)}{ab - h^2} + \frac{f(hg - af)}{ab - h^2} + c = 0. \tag{1}$$

The absolute term in this equation is

$$\frac{abc + 2fgh - bg^2 - af^2 - ch^2}{ab - h^2}. \tag{2}$$

For simplicity, let Δ represent the numerator in (2) and C, the denominator. Equation (1) can then be written

$$ax^2 + 2hxy + by^2 + \frac{\Delta}{C} = 0. \tag{3}$$

Δ can be expressed as a determinant. Thus,

$$\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}. \tag{4}$$

C is the cofactor corresponding to the element c, and the numerators in (3), Art. 113, are respectively the cofactors corresponding to the elements g and f. The determinant Δ is called the **discriminant** of the general equation of the second degree.

If the axes are now rotated so as to remove the term in xy, (3) becomes

$$a'x'^2 + b'y'^2 + \frac{\Delta}{C} = 0, \tag{5}$$

where a' and b' are the roots of (14), Art. 112.

If $ab - h^2 > 0$, a' and b' are alike in sign (Art. 112) and (5) can be written

$$|a'|x'^2 + |b'|y'^2 = c', \tag{6}$$

where $|a'|$ and $|b'|$ are positive numbers and c' is $\pm\frac{\Delta}{C}$. The locus is then an ellipse which is real or imaginary according as c' is positive or negative.

If $ab - h^2 < 0$, a' and b' are unlike in sign and then (5) can be written

$$|a'|x'^2 - |b'|y'^2 = c'. \tag{7}$$

The locus is then an hyperbola which is primary or conjugate with respect to the axes X', Y' according as c' is positive or negative.

Degenerate conics. If $\Delta = 0$, then c' is zero, and the locus of (6) is a single point; namely, the origin, and the locus of (7) is a pair of straight lines intersecting at the origin.

Neither a' nor b' can equal zero, since $a' \cdot b' = ab - h^2 \neq 0$. We conclude, therefore, that in this first case, the locus of the general equation of second degree is either an ellipse, real or imaginary; an hyperbola; a pair of real and intersecting lines; or a single point. *But the locus is never a parabola, a pair of parallel lines, or a single line.*

We may further conclude, from equations (3), (6), or (7), that the locus, whatever it may be, is symmetrical with respect to the origin, since if x_1 and y_1 satisfy any one of these equations, $-x_1$ and $-y_1$ will also satisfy the equation. Therefore the locus of the general equation, in this first case, is symmetrical with respect to the point (m, n). Or, in other words, the point (m, n) is the center of the locus. Hence the condition $ab - h^2 \neq 0$ characterizes the *central conics.*

As an example of the foregoing analysis, let us reduce the equation

$$8x^2 + 4xy + 5y^2 + 8x - 16y - 16 = 0$$

to the standard form and thus determine the nature and position of the locus.

Here $C = ab - h^2 = 40 - 4 = 36$, and $\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = -1296.$

Also from (3), Art. 113, we have

$$m = -1 \text{ and } n = 2.$$

Therefore the center of the conic is the point $(-1, 2)$. Again, from (9), Art. 112,

$$\tan 2\theta = \tfrac{4}{3},$$

from which we find $\theta = 26° 34'$, nearly. Hence we conclude that when the axes are translated so that the new origin is the point $(-1, 2)$, the given equation will take the form (3), or

$$8x^2 + 4xy + 5y^2 - 36 = 0,$$

and when the axes are rotated through the angle 26° 34′, the equation will take the form (5), where a' and b' are the roots of

$$\lambda^2 - 13\lambda + 36 = 0;$$

i.e. 9 and 4. Since h is positive, we choose $a' = 9$ and $b' = 4$. The given equation then becomes

$$9x^2 + 4y^2 - 36 = 0 \quad \text{or} \quad \frac{x^2}{4} + \frac{y^2}{9} = 1.$$

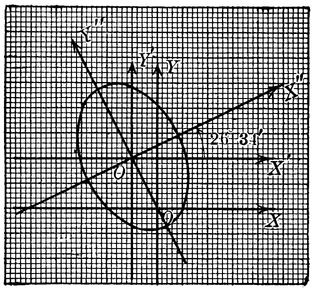

FIG. 90

The locus is therefore an ellipse whose semiaxes are 2 and 3. Figure 90 shows the curve and the three sets of axes.

As a second example consider the equation

$$2x^2 - xy - 3y^2 - 2x + 18y - 24 = 0.$$

Here we find $\Delta = 0$ and $C = -\frac{25}{4}$. The roots of (14), Art. 112, are therefore unlike in sign, and the locus consists of two intersecting lines. The left-hand member of the given equation must be the product of two linear factors

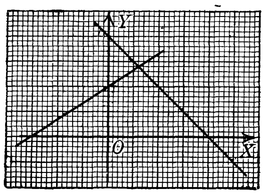

FIG. 91

(Art. 85). Solving the given equation for one of the variables, considering the other as a known number, reveals the factors at once. Thus, solving for x, we have

$$x = \frac{y+2}{4} \pm \frac{5y-14}{4}.$$

Hence the locus consists of the two lines

$$2x - 3y + 6 = 0 \text{ and } x + y - 4 = 0 \quad \text{(Fig. 91)}.$$

EXERCISES

1. Reduce the following equations to standard form. Determine the coördinates of the center and the angle through which the axes must be rotated in order to remove the term in xy:

(*a*) $5x^2 + 4xy - y^2 + 24x - 6y - 5 = 0$.
(*b*) $xy + y^2 + y + 1 = 0$.
(*c*) $4xy + 4y^2 - 2x + 3 = 0$.
(*d*) $x^2 + xy + y^2 + 3y = 0$.
(*e*) $x^2 - 2xy + 5y^2 - 8y = 0$.
(*f*) $x^2 + 2xy + 9y^2 = 0$.
(*g*) $2x^2 - 6xy + 5y^2 + 6x - 12y + 9 = 0$.

115. Second case, $ab - h^2 = 0$ and $hf - bg \neq 0$. Non-central conics. In this case the terms of first degree cannot be removed, since equations (2), Art. 113, are inconsistent and have no common solution. We begin the discussion, therefore, by rotating the axes so as to remove the term in xy. The angle through which the axes must be rotated is determined from (9), Art. 112. After rotation, the general equation assumes the form (8), Art. 112, where a' and b' are the roots of (14) and g' and f' are determined from (6) and (7). But in the case we are discussing, $ab - h^2 = 0$ and equation (14) becomes

$$\lambda^2 - (a + b)\lambda = 0$$

whose roots are 0 and $a + b$. We may assume that a is positive (Art. 111) then b is also positive, since the product ab is positive ($= h^2$). Hence we choose $a' = a + b$, or $b' = a + b$, according as h is positive or negative (Art. 112). The general equation is then reducible to one or the other of the forms

$$(a + b)x'^2 + 2g'x' + 2f'y' + c = 0, \tag{1}$$

or

$$(a + b)y'^2 + 2g'x' + 2f'y' + c = 0, \tag{2}$$

according as h is positive or negative.

We must now determine g' and f'. From (9), Art. 112, we have

$$\tan 2\theta \equiv \frac{2\tan\theta}{1-\tan^2\theta} = \frac{2h}{a-b}.$$

Solving this quadratic for $\tan\theta$, we obtain

$$\tan\theta = -\frac{a-b}{2h} \pm \sqrt{\frac{(a-b)^2}{4h^2}+1}.$$

But since $h^2 = ab$, the expression under the radical is a perfect square. Therefore, $\tan\theta = \frac{b}{h}$ or $-\frac{a}{h}$, from which $\sin\theta$ and $\cos\theta$, and thence g' and f' can be calculated. But, since $\theta < 90°$, $\tan\theta$, $\sin\theta$, and $\cos\theta$ are all positive. Hence we have the following results, where the sign before the radical is positive:

	h positive	h negative	
$\tan\theta =$	$\frac{b}{h}$	$-\frac{a}{h}$	
$\sin\theta =$	$\frac{b}{\sqrt{b^2+h^2}}$	$\frac{a}{\sqrt{a^2+h^2}}$	
$\cos\theta =$	$\frac{h}{\sqrt{b^2+h^2}}$	$\frac{-h}{\sqrt{a^2+h^2}}$	
$g' =$	$\frac{hg+bf}{\sqrt{b^2+h^2}}$	$\frac{af-hg}{\sqrt{a^2+h^2}}$	(3)
$f' =$	$\frac{hf-bg}{\sqrt{b^2+h^2}}$	$\frac{-(hf+ag)}{\sqrt{a^2+b^2}}$	(4)

Since neither $hf - bg$ nor $hg - af$ is zero, we see that if h is positive, and therefore the equation reducible to the form (1), f' cannot equal zero; and if h is negative, and the equation reducible to the form (2), g' cannot equal zero. Hence, on comparison with equations (3) and (4), Art. 111, we conclude that *the locus of the general equation, in this case, is necessarily a parabola.*

As an example, let us reduce the equation

$$x^2 - 2\,xy + y^2 - 2\,y - 1 = 0$$

to the standard form and thus determine the position of the locus.

Since $ab - h^2$ is here equal to zero and hf is not equal to bg, the locus is a parabola. Again, since h is negative, we choose the form (2). Computing g' and f' from (3) and (4), the given equation becomes

$$2\,y'^2 - \sqrt{2}\,x' - \sqrt{2}\,y' - 1 = 0.$$

Completing the square of the terms in y', we have

$$\left(y' - \frac{1}{2\sqrt{2}}\right)^2 = \frac{\sqrt{2}}{2}\left(x' + \frac{5}{4\sqrt{2}}\right).$$

Comparing with (5), Art. 110, we see that the vertex of the parabola, referred to the new axes, is the point $\left(-\frac{5}{4\sqrt{2}}, \frac{1}{2\sqrt{2}}\right)$. If the axes are translated so that this point is the new origin, the equation reduces to the standard form

$$y^2 = \frac{\sqrt{2}}{2}x,$$

where the primes have been dropped.

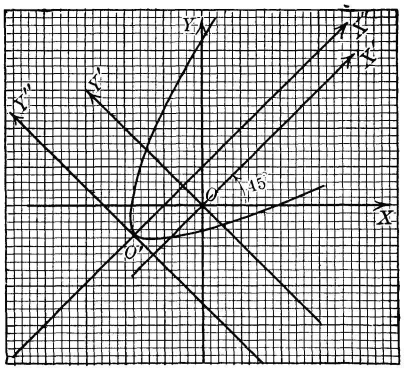

Fig. 92

The angle through which the axes have been rotated is given by the equation

$$\tan\theta = -\frac{a}{h} = 1.$$

Therefore $\theta = 45°$. Figure 92 shows the locus and the three sets of axes.

EXERCISES

1. Reduce the following equations to the standard form. Determine the angle through which it is necessary to rotate the axes in order to remove the term in xy, and the coördinates of the vertex referred to the original axes:

(*a*) $x^2 - 2\,xy + y^2 - 8\,x + 16 = 0.$
(*b*) $x^2 - 2\,xy + y^2 + 2\,x - y - 1 = 0.$
(*c*) $4\,x^2 + 4\,xy + y^2 - 4\,x = 0.$
(*d*) $9\,x^2 + 12\,xy + 4\,y^2 - 2\,y = 0.$
(*e*) $x^2 + 4\,xy + 4\,y^2 - 6\,x + 8\,y + 1 = 0.$

116. Third case, $ab - h^2 = 0$ and $hf - bg = 0$. Since $ab - h^2$ is again equal to zero, the general equation is reducible to the form (1), or the form (2), of the preceding article, according as h is positive or negative. But here $hf - bg = 0$ and $af - hg = 0$. Hence, if h is positive, $f' = 0$; and if h is negative, $g' = 0$. Consequently equations (1) and (2) of the preceding article become respectively

$$(a + b)x'^2 + 2\,g'x' + c = 0, \qquad (1)$$

and

$$(a + b)y'^2 + 2\,f'y' + c = 0. \qquad (2)$$

Each of these equations contains but a single variable. Therefore, in this case (cf. Art. 111), *the locus consists of a pair of parallel lines; a single line; or a pair of imaginary lines, according as the roots of the equation,* (1) *or* (2), *are real and distinct; equal; or imaginary.*

It is not necessary to calculate the coefficients g' and f' in order to determine the nature of the locus of an equation satisfying the conditions of this third case. For, since a is not zero, the general equation of the second degree can be written

$$a^2x^2 + 2\,ahxy + aby^2 + 2\,agx + 2\,afy + ac = 0, \qquad (3)$$

and since $ab = h^2$ and $af = hg$, (3) becomes

$$(ax + hy)^2 + 2\,g(ax + hy) + ac = 0. \qquad (4)$$

The locus then consists of a pair of parallel lines, a single line, or a pair of imaginary lines according as g^2 is greater than, equal to, or less than ac. For example, the locus of the equation

$$4\,x^2 + 12\,xy + 9\,y^2 + 4\,x + 6\,y + 1 = 0$$

is a single line, since here $g^2 = ac$. The equation of the line is clearly

$$2\,x + 3\,y + 1 = 0.$$

EXERCISES

1. Determine the nature of the loci of the following equations. Draw the locus when possible.

(*a*) $x^2 - 2\,xy + y^2 + 2\,y - 2\,x + 1 = 0.$
(*b*) $4\,x^2 + 12\,xy + 9\,y^2 + 4\,x + 6\,y + 2 = 0.$
(*c*) $x^2 + 2\,xy + y^2 - 1 = 0.$
(*d*) $9\,x^2 - 12\,xy + 4\,y^2 + 15\,x - 10\,y + 6 = 0.$

2. If $a = 0$, in the third case, show that the general equation is necessarily $by^2 + 2fy + c = 0$, and therefore the locus is a pair of parallel lines, a single line, or a pair of imaginary lines according as f^2 is greater than, equal to, or less than bc.

117. Recapitulation. The results of the foregoing three articles can be exhibited in tabular form as follows:

Loci of the general equation of the second degree

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0.$$

$$\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}, \quad C = ab - h^2.$$

	First case, $ab - h^2 \neq 0$		Second case, $ab - h^2 = 0$, $hf - bg \neq 0$	Third case, $ab - h^2 = 0$, $hf - bg = 0$
	$C > 0$	$C < 0$		
$\Delta \neq 0$	Ellipse, real or imag.	Hyperbola	Parabolas	Parallel lines, a single line, or imaginary lines
$\Delta = 0$	Point	Intersecting lines		

EXERCISES

1. Analyze the following equations. What is the locus of each?

(a) $x^2 + 6xy + y^2 - 4x - 12y + 10 = 0$. (b) $x^2 - xy + y^2 + 3x = 0$.
(c) $9x^2 - 30xy + 25y^2 - 10x = 0$. (d) $2x^2 - xy + 5x - 2y + 6 = 0$.

2. Analyze each of the following equations and draw the corresponding locus.

(a) $x^2 - 2xy + y^2 - 10x - 6y + 25 = 0$. (b) $x^2 - xy + 5x - 2y + 6 = 0$.
(c) $2x^2 + xy + y^2 - 5x - 10y + 18 = 0$. (d) $x^2 + 3xy + 2y^2 - x - y = 0$.

3. The locus of the equation $3x^2 - 3xy - y^2 + 15x + 10y - 24 = 0$ is an hyperbola; find the equations of its asymptotes.

Suggestion. The center of the curve is found to be the point (0, 5). The standard form of the equation is $3x^2 - 7y^2 = 2$. Hence the equations of the asymptotes, referred to the axes of the curve, are $3x^2 - 7y^2 = 0$

(Art. 106). When the coördinate axes are transformed back to the original position, the equations of the asymptotes become

$$3x^2 - 3x(y-5) - (y-5)^2 = 0,$$

or

$$y - 5 = (-3 \pm \sqrt{21})\frac{x}{2}.$$

4. For what value of k is the locus of $x^2 + 2xy + 2y^2 + x + k = 0$ a pair of straight lines? Are these lines real or imaginary?

5. If the locus of the general equation of second degree in x and y is a central conic, show that the equation can be written in the form

$$a(x-m)^2 + 2h(x-m)(y-n) + b(y-n)^2 = -\frac{\Delta}{C},$$

where m and n are the coördinates of the center, and Δ and C have the meanings assigned in Art. 117.

6. Making use of the preceding exercise, show that the equations of the asymptotes of any hyperbola are

$$a(x-m)^2 + 2h(x-m)(y-n) + b(y-n)^2 = 0.$$

Apply this method to find the equations of the asymptotes of the hyperbola

$$x^2 + 6xy + y^2 - 4x - 12y + 10 = 0.$$

7. Find the coördinates of the vertex, the coördinates of the focus, and the equation of the directrix of the parabola

$$x^2 - 4xy + 4y^2 - 4x - 2y + 8 = 0.$$

TANGENTS AND DIAMETERS

118. Tangents. It is often convenient to write the equation of a tangent to a conic at a given point without first having to reduce the equation to the standard form. Suppose the equation has the general form

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0. \qquad (1)$$

Let $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ be any two points on the curve. Then we must have

$$ax_1^2 + 2hx_1y_1 + by_1^2 + 2gx_1 + 2fy_1 + c = 0, \qquad (2)$$

and

$$ax_2^2 + 2hx_2y_2 + by_2^2 + 2gx_2 + 2fy_2 + c = 0. \qquad (3)$$

Subtracting (3) from (2), we obtain

$$a(x_1^2 - x_2^2) + 2h(x_1y_1 - x_2y_2) + b(y_1^2 - y_2^2) + 2g(x_1 - x_2) + 2f(y_1 - y_2) = 0. \qquad (4)$$

Dividing (4) by $x_1 - x_2$, we have

$$a(x_1+x_2)+2h\frac{x_1y_1-x_2y_2}{x_1-x_2}+b\frac{(y_1+y_2)(y_1-y_2)}{x_1-x_2}+2g+2f\frac{y_1-y_2}{x_1-x_2}=0. \quad (5)$$

Now $\frac{y_1-y_2}{x_1-x_2}$ is the slope of the secant P_1P_2. Let this slope be represented by m. The term $\frac{x_1y_1-x_2y_2}{x_1-x_2}$ can be written

$$\frac{x_1y_1-x_1y_2+x_1y_2-x_2y_2}{x_1-x_2},$$

and is therefore equal to mx_1+y_2. Hence (5) becomes

$$a(x_1+x_2)+2h(mx_1+y_2)+b(y_1+y_2)m+2g+2fm=0. \quad (6)$$

Solving for m, we obtain

$$m=-\frac{a(x_1+x_2)+2hy_2+2g}{2hx_1+b(y_1+y_2)+2f}. \quad (7)$$

Equations (2) to (7) hold as long as P_1 and P_2 are on the curve. When P_2 approaches P_1 along the curve, the secant approaches the position of the tangent at P_1, and in the limit coincides with it (Art. 97, second method). Hence, making $x_2=x_1$ and $y_2=y_1$ in (7), we have the slope of the tangent at P_1; that is,

$$\boldsymbol{m=-\frac{ax_1+hy_1+g}{hx_1+by_1+f}}. \quad (8)$$

The equation of the tangent at P_1 is, therefore,

$$y-y_1=-\frac{ax_1+hy_1+g}{hx_1+by_1+f}(x-x_1).$$

Clearing of fractions and reducing by means of equation (2), we have

$$ax_1x+h(x_1y+y_1x)+by_1y+g(x_1+x)+f(y_1+y)+c=0. \quad (9)$$

This equation is easily remembered, since if the subscripts are removed, it returns to the original form (1).

A convenient way of writing (9) is the following:

$$\begin{aligned} &x(ax_1+hy_1+g)\\ &+y(hx_1+by_1+f)\\ &+(gx_1+fy_1+c)=0. \end{aligned} \quad (10)$$

Either (9) or (10) is the equation of a straight line whether the point P_1 is on the conic or not. If P_1 is not on the conic, then (9) or (10) is, by definition (Art. 104), the equation of the polar line of P_1 with respect to the conic whose equation is (1).

EXERCISES

1. Find the equations of the tangents to the following, at the points indicated.

(*a*) $x^2 + 4y^2 + 5x = 0$, at the points whose ordinate is 1.
(*b*) $xy = 4$, at the point whose abscissa is 2.
(*c*) $x^2 + xy + 4 = 0$, at the point whose abscissa is 2.
(*d*) $y^2 + 2xy - 3 = 0$, at the point whose ordinate is -1.
(*e*) $x^2 - 3xy - 4y^2 + 9 = 0$, at the points whose ordinate is 2.

2. Find the equation of the polar line of the point (1, 2) with respect to the conic $x^2 - 3xy + y^2 = 4$. Draw the figure to illustrate the problem.

3. In exercise 7, Art. 117, show that the directrix is the polar line of the focus with respect to the given parabola.

119. Diameters. In case of a central conic, we have found that the coördinates of the center are the values of m and n which satisfy equations (2), Art. 113. This amounts to saying that the center of the conic is the point of intersection of the lines whose equations are

$$\begin{aligned} ax + hy + g &= 0, \\ hx + by + f &= 0. \end{aligned} \tag{1}$$

Hence these two lines are diameters of the conic (Art. 101). The equation of any diameter is, therefore (Art. 84),

$$(ax + hy + g) + k(hx + by + f) = 0 \tag{2}$$

where k is a variable parameter.

Let $P(x_1, y_1)$ be any point on the conic. When the diameter (2) passes through P, k has the value

$$-\frac{ax_1 + hy_1 + g}{hx_1 + by_1 + f}.$$

But this is the slope of the tangent at P, (8), Art. 118, and hence, also the slope of the diameter conjugate to (2), Art. 102. Therefore, *the parameter* k *is the slope of the diameter conjugate to* (2). The slope of (2) is

$$-\frac{a+hk}{h+bk}.$$

Hence, k *and* $-\dfrac{a+hk}{h+bk}$ *are the slopes of a pair of conjugate diameters of the conic whose equation has the general form* (1) *of the preceding article.*

The two diameters will be perpendicular to each other, and therefore the axes (Art. 102, exercises 12 and 13), when the product of their slopes is -1; that is, when $\dfrac{ak+hk^2}{h+bk}=1$, or

$$\boldsymbol{hk^2+(a-b)k-h=0.} \qquad (3)$$

If k_1 and k_2 are the roots of (3), the equations of the axes are

$$\left.\begin{aligned}&\boldsymbol{(ax+hy+g)+k_1(hx+by+f)=0,}\\&\boldsymbol{(ax+hy+g)+k_2(hx+by+f)=0.}\end{aligned}\right\} \qquad (4)$$

The roots of (3) are always real, since the discriminant,

$$4\,h^2+(a-b)^2,$$

is necessarily positive.

We have seen (Art. 106) that an asymptote of an hyperbola is a self-conjugate diameter. But if the slope of any diameter of a conic is equal to the slope of its conjugate diameter, we must have

$$k=-\frac{a+hk}{h+bk},$$

or

$$\boldsymbol{bk^2+2\,hk+a=0.} \qquad (5)$$

If k' and k'' are the roots of (5), the equations of the asymptotes are

$$\boldsymbol{(ax+hy+g)+k'(hx+by+f)=0,}$$

and

$$\boldsymbol{(ax+hy+g)+k''(hx+by+f)=0.} \qquad (6)$$

The roots of (5) are real and unequal only if $ab-h^2<0$; that is, only if the conic is an hyperbola or a pair of intersecting lines (Art. 117).

As an example, consider the equation

$$2\,x^2+4\,xy-y^2+4\,x-2\,y+3=0.$$

Here equation (3) is

$$2\,k^2+3\,k-2=0,$$

whose roots are $\frac{1}{2}$ and -2. Hence, from (4), the equations of the axes are

$$2x+y+1=0,$$

and

$$x-2y-2=0.$$

Equation (5) becomes in this case

$$k^2-4k-2=0,$$

whose roots are $2\pm\sqrt{6}$. Hence the equations of the asymptotes are

$$(2x+2y+2)+(2\pm\sqrt{6})(2x-y-1)=0.$$

The student should draw a figure illustrating this example.

The diameters of a parabola are perpendicular to the directrix (Art. 101) and therefore perpendicular to the tangent to the curve at the vertex. We have seen (Art. 115) that the equation of a parabola is reducible to one or the other of two forms by a rotation of the axes through an angle $\theta < 90°$. But if h is positive, the new X-axis is parallel to the tangent at the vertex (Eq. 1, Art. 115); and if h is negative, the new X-axis is parallel to the axis of the curve (Eq. 2, Art. 115). Hence, from the values of $\tan\theta$ in these two cases, we conclude that *the slope of a diameter to the parabola is* $-\frac{h}{b}$ *or* $-\frac{a}{h}$ *according as h is positive or negative.* But $\frac{h}{b}=\frac{a}{h}\cdot$ Consequently, $-\frac{h}{b}$ *is the slope of any diameter.*

The slope of the tangent to the parabola at the point $P(x, y)$ is, (8), Art. 118,

$$m=-\frac{ax+hy+g}{hx+by+f}\cdot$$

This tangent is perpendicular to the diameters of the curve if $\frac{mh}{b}=1$, or in other words, if the coördinates of the point of contact satisfy the equation

$$\frac{(ax+hy+g)h}{(hx+by+f)b}=-1. \tag{7}$$

Clearing of fractions and remembering that $h^2=ab$, (7) becomes

$$\boldsymbol{(a+b)(hx+by)+hg+fb=0.} \tag{8}$$

Now (8) is the equation of the axis of the parabola. For it is a diameter of the curve since it has the slope $-\frac{h}{b}$.[*]

As an example, consider the equation

$$x^2 - 2\,xy + y^2 - 2\,y - 1 = 0. \quad \text{(cf. Art. 115.)}$$

From (8), we get the equation of the axis,

$$2(x - y) + 1 = 0.$$

If we solve the equation of the curve and the equation of the axis simultaneously, we get the coördinates of the vertex. In this example, the vertex is the point $(-\frac{7}{8}, -\frac{3}{8})$. The equation of the tangent at the vertex is, therefore,

$$y + \tfrac{3}{8} = -(x + \tfrac{7}{8}), \text{ or } x + y + \tfrac{5}{4} = 0.$$

As a second example, consider the equation

$$9\,x^2 + 24\,xy + 16\,y^2 - 52\,x + 14\,y - 6 = 0.$$

Here we find the equation of the axis is

$$3\,x + 4\,y - 2 = 0.$$

Solving simultaneously with the given equation, we find that the vertex is the point (.08, .44). The equation of the tangent at the vertex is, therefore,

$$(y - .44) = \tfrac{4}{3}(x - .08)$$

which reduces to

$$4\,x - 3\,y + 1 = 0.$$

The student should construct a figure to illustrate this example.

EXERCISES

1. Find the equations of the axes of the ellipse

$$x^2 - 2\,xy + 4\,y^2 + 2\,x + 10\,y + 10 = 0.$$

2. Find the equations of the axes and the equations of the asymptotes of the hyperbola

$$x^2 - 7\,xy + y^2 + 12\,x + 3\,y + 171 = 0.$$

3. Find the equation of the axis, of the tangent at the vertex, and of the directrix of the parabola

$$x^2 - 2\,xy + y^2 - 10\,x - 6\,y + 25 = 0.$$

4. In the general equation of a conic, show that the line $gx + fy + c = 0$ is the polar line of the origin with respect to the conic.

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[*] That the tangent at the vertex is the only tangent perpendicular to the diameter through its point of contact follows from Art. 96. We there saw that the coördinates of the point of contact are $\frac{p}{m^2}$ and $\frac{2p}{m}$, where m is the slope of the tangent. Hence, as m increases indefinitely, the point of contact approaches the origin.

SYSTEMS OF CONICS

120. The pencil of conics. *If U and V denote expressions of the second degree in x and y and k is any constant, then $U + kV = 0$ is the equation of a conic passing through the points common to $U = 0$ and $V = 0$.*

For $U + kV = 0$ is of second degree in x and y and is, therefore, the equation of a conic. This conic passes through the points common to $U = 0$ and $V = 0$, since its equation is satisfied by the coördinates of these points. When k is allowed to vary, we obtain a series of conics, each passing through the common points. This series, or system, of conics is called a **pencil of conics.**

The parameter k can be chosen so that the conic $U + kV = 0$ shall satisfy some additional condition, for example, that it shall pass through a given point in the plane.

121. The system of circles with a common radical axis. Suppose that $U = 0$ and $V = 0$ are the equations of two circles; that is,

$$U \equiv x^2 + y^2 + Ax + By + C = 0,$$

and

$$V \equiv x^2 + y^2 + A_1x + B_1y + C_1 = 0, \qquad (1)$$

then

$$(x^2 + y^2 + Ax + By + C) + k(x^2 + y^2 + A_1x + B_1y + C_1) = 0 \qquad (2)$$

is in general the equation of a circle passing through the common points of the two given circles. But if $k = -1$, the terms of second degree drop out, and (2) becomes

$$(A - A_1)x + (B - B_1)y + (C - C_1) = 0, \qquad (3)$$

which is of first degree in x and y, and therefore the equation of a straight line. This line is called the **radical axis** of the system of circles $U + kV = 0$. The radical axis is a real line, whether the circles intersect in real points or not. If the circles intersect in real points, the radical axis is the **common chord.**

EXERCISES

1. Find the equation of the conic which passes through the points common to the conics $x^2 - 3xy + y^2 - 6x = 0$ and $4x^2 - y^2 + 3 = 0$, and also through the point $(3, -2)$.

2. Find the equation of the radical axis of each of the following pairs of circles:

(*a*) $(x-2)^2+(y-3)^2-10=0,\ (x+3)^2+(y+2)^2-6=0.$

(*b*) $x^2+y^2-4y=0,\ (x-3)^2+y^2-9=0.$

(*c*) $(x+3)^2+y^2-2y-8=0,\ x^2+y^2-2y=0.$

(*d*) $x^2+(y-a)^2=c^2,\ (x-2)^2+y^2=d^2.$

3. Three circles, taken in pairs, have three radical axes. Show that these radical axes intersect in one and the same point. This point is called the **radical center** of the three circles.

4. Find the coördinates of the radical center of the three circles $(x-3)^2+y^2=16$, $x^2+y^2=9$, and $x^2+(y-2)^2=25$. Construct the figure illustrating this exercise.

5. Show that the length of a tangent from the point (x_1, y_1) to the point of contact on the circle $x^2+y^2+Dx+Ey+F=0$ is

$$\sqrt{x_1^2+y_1^2+Dx_1+Ey_1+F}.$$

SUGGESTION. The triangle whose vertices are the center of the circle, the point of contact, and the point (x_1, y_1) is right-angled at the point of contact.

6. Prove that the locus of a point, the lengths of tangents from which to two fixed circles are equal, is the radical axis of the two circles.

7. Show that the radical axis of two circles is perpendicular to the line joining the centers of the circles.

122. The parabolas in the pencil $U+kV=0$. If the constant k is chosen so that the terms of second degree in $U+kV=0$ form a perfect square, the corresponding conic is in general a parabola. But it may be a pair of parallel lines, a pair of imaginary lines, or a single line (Arts. 115 and 116). Since the condition for the parabola is of second degree in the coefficients of x^2, xy, and y^2, there are in general two parabolas in every pencil of conics.

For example, consider the pencil of conics determined by the circle
$$U\equiv x^2+y^2-16x-8y+44=0,$$
and the hyperbola
$$V\equiv x^2-y^2-8x+12=0.$$
Here the equation
$$(x^2+y^2-16x-8y+44)+k(x^2-y^2-8x+12)=0$$
can be the equation of a parabola only if $k=\pm 1$. Therefore the pencil contains the two parabolas
$$x^2-12x-4y+28=0 \text{ and } y^2-4x-4y+16=0.$$

Fig. 93 illustrates the example. The circle and hyperbola have but two real points in common and the two parabolas pass through these points.

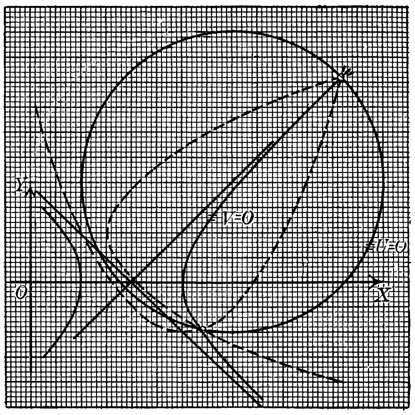

FIG. 93

EXERCISES

1. Find the equations of the parabolas which pass through the points common to the circle $x^2 + y^2 - x - 9 = 0$ and the hyperbola $xy - 1 = 0$.

2. Find the equations of the two parabolas which pass through the points where the ellipse $x^2 - 3xy + 4y^2 - x - 2 = 0$ cuts the coördinate axes. (The equation of the coördinate axes is $xy = 0$.) Construct the figure illustrating this exercise.

123. Straight lines in the pencil $U + kV = 0$. When k is chosen so that the discriminant of $U + kV = 0$ vanishes, the corresponding conic is in general a pair of lines (Art. 114).

For example, consider the pencil of conics determined by the ellipses

$$U \equiv 2x^2 + xy + 6y^2 + x - 6 = 0 \text{ and } V \equiv 3x^2 + 5xy + 10y^2 + x - 10 = 0.$$

Here the pencil $U + kV = 0$ is

$$(2 + 3k)x^2 + (1 + 5k)xy + (6 + 10k)y^2 + (1 + k)x - (6 + 10k) = 0.$$

Forming the discriminant,

$$\Delta = \begin{vmatrix} (2 + 3k) & \dfrac{1 + 5k}{2} & \dfrac{1 + k}{2} \\ \dfrac{1 + 5k}{2} & (6 + 10k) & 0 \\ \dfrac{1 + k}{2} & 0 & -(6 + 10k) \end{vmatrix},$$

we find that it reduces to

$$-12(6 + 10k)(k + 1)(2k + 1).$$

Hence, if k is $-\frac{3}{5}$, -1, or $-\frac{1}{2}$, the discriminant is zero and the corresponding conic consists of a pair of lines. If $k = -\frac{3}{5}$, $U + kV = 0$ becomes

$$x^2 - 10xy + 2x = 0,$$

which is the equation of the pair of lines

$$x = 0 \text{ and } x - 10\,y + 2 = 0.$$

These are the lines BC and AD in Fig. 94. If $k = -1$, $U + kV = 0$ becomes

$$x^2 + 4\,xy + 4\,y^2 - 4 = 0,$$

which is the equation of the pair of parallel lines

$$x + 2\,y - 2 = 0 \text{ and } x + 2\,y + 2 = 0,$$

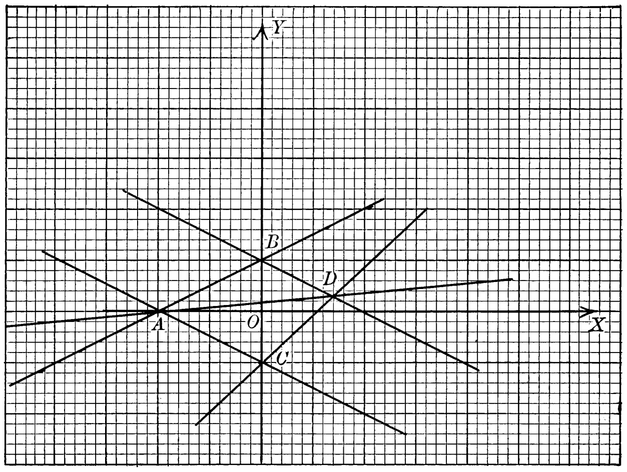

Fig. 94

or BD and AC in the figure. If $k = -\frac{1}{2}$, $U + kV = 0$ becomes

$$x^2 - 3\,xy + 2\,y^2 + x - 2 = 0,$$

or

$$(x - 2\,y + 2)(x - y - 1) = 0,$$

which is the equation of the pair of lines AB and CD.

The coördinates of the points A, B, C, and D are now easily found. They represent the common solutions of the equations $U = 0$ and $V = 0$.

EXERCISES

1. Find the equations of the straight lines which join in pairs the points common to the following pair of conics:

(*a*) $x^2 + y^2 - 25 = 0,\ 5\,x^2 + 14\,y + 3\,x - 110 = 0.$

(*b*) $4\,x^2 + 9\,y - 36 = 0,\ x^2 + 4\,y = 0.$

(*c*) $x^2 + 2\,xy + 7\,y^2 - 24 = 0,\ 2\,x^2 - xy - y^2 - 8 = 0.$

2. Find the coördinates of the points common to each pair of conics in exercise 1.

124. The pencil of conics through four given points. In the preceding article we have seen how the coördinates of the points common to two conics $U=0$ and $V=0$ may be found. On the other hand, if we are given the coördinates of four points, we can determine the pencil of conics which has these points in common.

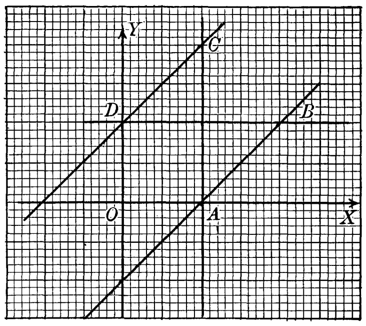

FIG. 95

For example, let the four given points be $A(1, 0)$, $B(2, 1)$, $C(1, 2)$, and $D(0, 1)$ (Fig. 95). The equation of the pair of lines AB and CD is then

$$(x - y - 1)(x - y + 1) = 0,$$

and the equation of the pair AC and BD is

$$(x-1)(y - 1) = 0.$$

Therefore the equation of the pencil of conics having the four given points in common is

$$(x - y - 1)(x - y + 1) + k(x - 1)(y - 1) = 0.$$

Clearly, the parameter k can be determined so that the corresponding conic shall pass through any fifth point in the plane. Thus, if we wish the conic of the pencil which passes through the origin, we must determine k so that the above equation shall be satisfied by the coördinates of the origin. But the equation is satisfied for $x = 0$ and $y = 0$ if k is 1. Therefore the ellipse

$$x^2 - xy + y^2 - x - y = 0$$

belongs to the pencil and also passes through the origin.

EXERCISES

1. Find the equations of the conics which pass through the following sets of points:

(a) (0, 0), (1, 0), (2, 1), (1, 3), and (– 1, – 4).
(b) (1, 1), (3, 2), (0, 4), (– 4, 0), and (– 2, – 2).

MISCELLANEOUS EXERCISES

1. Show that the line $x - 2y = 0$ touches the circle

$$x^2 + y^2 - 4x + 8y = 0.$$

2. The line $y = 3x - 9$ is tangent to the circle

$$x^2 + y^2 + 2x + 4y - 5 = 0.$$

Find the coördinates of the point of contact.

3. Prove that the distances of two points from the center of a circle are proportional to the distances of each from the polar line of the other.

4. Find the equations of the circles which pass through the intersections of

$$x^2 + y^2 = 9 \text{ and } x^2 + y^2 + x + 2y = 14$$

and touch the X-axis.

5. Find the coördinates of two points whose polar lines with respect to the circles $x^2 + y^2 - 2x - 3 = 0$ and $x^2 + y^2 + 2x - 17 = 0$ coincide.

6. Find the coördinates of the radical center of the three circles

$$x^2 + y^2 - 4x - 8y - 5 = 0,\ x^2 + y^2 - 8x - 10y + 25 = 0,$$

and

$$x^2 + y^2 + 8x + 11y - 10 = 0.$$

7. Reduce the following equations to a standard form:

(*a*) $(4y - 3x)^2 + 4(4x + 3y) = 0.$
(*b*) $4x^2 - 24xy + 11y^2 - 16x - 2y - 89 = 0.$
(*c*) $5x^2 - 4xy + 8y^2 - 22x + 16y - 10 = 0.$
(*d*) $9x^2 - 12xy + 4y^2 = 10(2x + 3y + 5).$
(*e*) $3x^2 - 2xy + 2y^2 - 16x - 8y + 8 = 0.$
(*f*) $6x^2 + 24xy - y^2 + 50y - 55 = 0.$
(*g*) $x^2 - 2xy + y^2 - 5x - y - 2 = 0.$
(*h*) $4x^2 + 4xy + y^2 + 4x - 3y + 4 = 0.$
(*i*) $25x^2 - 20xy + 4y^2 + 5x - 2y - 6 = 0.$
(*j*) $x^2 - 6xy + 9y^2 - 2x + 6y + 1 = 0.$
(*k*) $x^2 - 2xy - y^2 = 20.$
(*l*) $xy + 3x - 5y + 5 = 0.$
(*m*) $x^2 + 2xy + y^2 + 1 = 0.$
(*n*) $(5y + 12x)^2 = 102x.$
(*o*) $x^2 - xy - 2y^2 - x - 4y - 2 = 0.$

8. What curve must be used as a pattern for cutting elbows of stovepipes from sheet iron?

CHAPTER IX

LOCI OF HIGHER ORDER AND OTHER LOCI

125. Certain loci of higher order, as well as certain transcendental loci, are of importance either because they are useful in mechanics or because of their historical interest. The more important of these loci are considered in the following articles.

ALGEBRAIC LOCI

126. The Cissoid of Diocles. Let C be the center of a circle of radius a, and OCA, any diameter of it. Through O draw any chord OR and produce it until it meets the tangent at A in the point Q. If P is so chosen that PQ is equal to OR, then the locus of P is a curve called the **Cissoid of Diocles.**

To find the equation of the cissoid, let O be the origin, OCA the X-axis, and the tangent at O the Y-axis. Let θ denote the angle AOQ. Then $OQ = 2a \sec\theta$ and $OR = 2a\cos\theta$. Hence,

$$OP = OQ - PQ = OQ - OR = 2a(\sec\theta - \cos\theta). \qquad (1)$$

But $$OP = \sqrt{x^2+y^2} \text{ and } \theta = \arc\tan\frac{y}{x}.$$

Therefore $$\sec\theta = \frac{\sqrt{x^2+y^2}}{x},$$

and $$\cos\theta = \frac{x}{\sqrt{x^2+y^2}}.$$

Substituting in (1) and reducing, we get the equation sought,

$$y^2 = \frac{x^3}{2a-x}.$$

Either from the definition, or the equation, the curve is seen to have the form indicated in the figure.

EXERCISES

1. Show that the line $2a - x = 0$ is an asymptote to the cissoid.

2. Using the method of Art. 118, show that the tangent to the cissoid at the point (x_1, y_1) is $2(2a - x_1)y_1y - (3x_1^2 + y_1^2)x + 2ay_1^2 = 0$.

3. In Fig. 96 let CM be taken twice the length of CB. Draw MA and let it meet the cissoid in the point F whose ordinate is FE. Prove that $FE^3 = 2 \cdot OE^3$. If CM is n times CB, show that $FE^3 = n \cdot OE^3$.

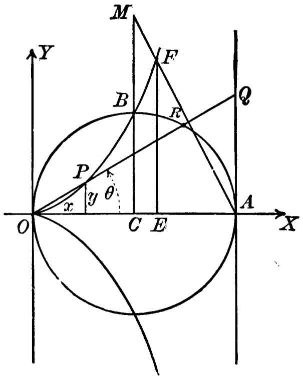

FIG. 96

NOTE. The cissoid was invented by Diocles for the purpose of duplicating the cube. Thus, in Fig. 96, when CM is twice CB, and OE is the edge of a given cube, FE is the edge of a cube of twice the volume. The duplication of the cube is one of the famous problems of antiquity. Diocles lived about 150 B.C.

127. The Conchoid of Nicomedes. Let XX' be any straight line and O any point not on XX'. Through O draw a series of straight lines forming a pencil, and on each of these lines lay off a constant length a on each side of XX'. The locus of the points so determined is called the **conchoid of Nicomedes.**

To find the equation of the conchoid, let XX' be the X-axis and the perpendicular through O, the Y-axis. The point of intersection A is the origin. Let $OA = b$, and P, any point on the conchoid. Construct the right triangle POD, PO and PD meeting XX' in F and E, respectively. From similar triangles, we have

$$EP : FE :: DP : OD.$$

Now, $EP = y$, $DP = y + b$, and $OD = x$. By construction, $PF = a$, and hence $FE = \sqrt{a^2 - y^2}$. It is clear that these relations hold also for the point P', where $P'F = FP = a$. Substituting in (1) and reducing, we have the equation sought; namely,

$$x^2y^2 = (y + b)^2(a^2 - y^2).$$

EXERCISES

1. Construct conchoids for which $a > b$, $a = b$ and $a < b$. Note the difference in form.

2. Show that the X-axis is an asymptote of the conchoid.

3. In Fig. 97, let AB be twice the length of OF. Draw the perpendicular XX' at F and let it meet the conchoid at K. Draw OK and take $KR = OF$. Show that $KR = RF = OF$, and consequently the angle KOB is one third the angle POB. Show how this construction enables one to trisect any given angle.

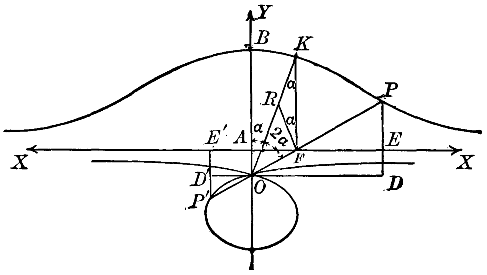

FIG. 97

NOTE. The conchoid was invented by Nicomedes for the purpose of trisecting a given angle. This is another famous problem of antiquity. Neither the duplication of the cube nor the trisection of an angle can be effected by means of the circle and straight line alone, hence the ancients were forced to the invention of other curves for these purposes. Nicomedes was a contemporary of Diocles.

128. The Witch of Agnesi. Let C be the center of a circle whose radius is a; and OB any diameter. Draw the tangents at O and B; and let OR be any chord through O which, produced, meets the tangent at B in N. Through R draw the parallel to OD, the X-axis, and through N, the parallel to OB, the Y-axis. The locus of the point P, where these parallels meet, is called the **witch of Agnesi**.

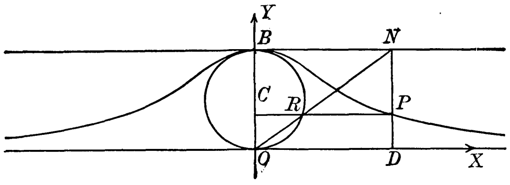

FIG. 98

EXERCISES

1. Show that the equation of the witch, referred to the lines OX and OY as coördinate axes, is $y = \frac{8a^3}{x^2 + 4a^2}$.

SUGGESTION. Use the similar triangles NRP and NOD to find the relation between the coördinates of P.

2. Show that the X-axis is an asymptote of the witch.

NOTE. Donna Maria Agnesi, who invented the witch, was born at Milan, 1718, and died there, 1799. She was appointed Professor of Mathematics at the University of Bologna, 1750.

129. The Limaçon of Pascal. Let C be the center of a circle whose radius is a; and OD, any diameter. Through O draw a series of lines, and on each of these lay off a distance b on each side of the circle. The locus of the points thus determined is called the **limaçon.**

To find the equation of the limaçon, let O be the pole and OD the polar axis. The length of the chord within the circle is $2a \cos\theta$. Hence the radii of the points P and P' on this chord are given by the equation

$$r = 2a\cos\theta \pm b,$$

which in rectangular coördinates reduces to

$$(x^2 + y^2 - 2ax)^2 = b^2(x^2 + y^2).$$

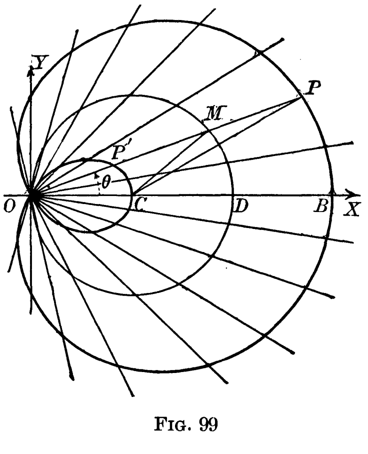

FIG. 99

EXERCISES

1. Construct the limaçons for which $b > 2a$, $b = 2a$, and $b < 2a$. Note the difference in form. When $b = 2a$, the limaçon is called the cardioid from its heart-shaped form.

2. When $b = a$, the limaçon furnishes a neat curve for trisecting a given angle. In Fig. 99, let PCB be the given angle. Show that $PM = MC = CO$ and, therefore, the angle POB is $\frac{1}{3}$ the angle PCB.

NOTE. Pascal (1623–1662) was a celebrated French mathematician and philosopher.

MISCELLANEOUS EXERCISES

1. Show that the locus of the intersection of a tangent to the parabola $y^2 = -8ax$ and a line drawn through the origin perpendicular to this tangent is the cissoid.

Suggestion. The equation of a tangent in terms of the slope is (Art. 95, Eq. 9) $y = mx - \frac{2a}{m}$, and the equation of the line through the origin perpendicular to this tangent is $y = -\frac{x}{m}$. The locus of the intersection of these two lines is found by eliminating m from the two equations.

2. A tangent is drawn to the parabola $y^2 = 4px$ at a point T. The perpendicular to this tangent through the origin meets the ordinate of T, produced, at P. Find the equation of the locus of P as T moves along the curve. The locus is called the **semicubical parabola**.

3. The two parabolas $y^2 = 2ax$ and $x^2 = ay$ meet at the origin and also at another point P. Find the coördinates of P. If a is the edge of a given cube, show how the construction of the two parabolas solves the problem of the duplication of the cube.

4. Show that the conchoid is the locus of the points of intersection of the line $y = \frac{x}{k} - b$ with the circle $(x - bk)^2 + y^2 = a^2$, k being a variable parameter.

5. A tangent is drawn to the equilateral hyperbola $x^2 - y^2 = a^2$ at the point T. The perpendicular to this tangent through the origin meets the tangent in the point P. Show that the locus of P, as T moves along the curve, is the lemniscate (Art. 54).

6. Find the locus of the intersection of the two straight lines

$$x + ky + a(k^2 - 3) = 0 \text{ and } y = kx,$$

k being a variable parameter. The locus is called the **trisectrix of Maclaurin.** Discuss its equation and draw the locus.

TRANSCENDENTAL LOCI

130. The cycloid. The locus of a point in the circumference of a circle which rolls (without sliding) along a fixed straight line is called the **cycloid**. The circle is called the **generating circle.**

To find the equation of the cycloid, take the line on which the circle rolls as the X-axis, the radius of the rolling circle equal to a, and one of the positions in which the tracing point is on this line as the origin O. Let C be the center of the generating circle when the tracing point has the position P. Join P and C, and draw CT perpendicular to the X-axis. Let θ represent the angle PCT. Now $OT = \text{arc } PT = a\theta$. Hence,

$$\begin{aligned} x &= OD = OT - PE = a\theta - a\sin\theta, \\ y &= DP = TC - EC = a - a\cos\theta. \end{aligned} \tag{1}$$

Equations (1) are the parametric equations of the cycloid, θ being the parameter.

When θ varies from 0 to 2π, P traces out one arch of the curve. The entire curve consists of this arch and repetitions of it to the right and left corresponding to values of θ outside the limits $0, 2\pi$.

EXERCISES

1. In Fig. 100, OB is the span of one arch of the cycloid and F is its middle point. Show that the area of the triangle OAB is twice the area of the generating circle.

2. When the point T bisects OF, show that the area of the triangle OPA is half the area of the square inscribed in the generating circle.

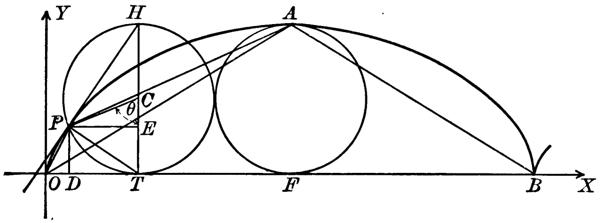

Fig. 100

3. Prove that the tangent to the cycloid at P passes through H, the upper extremity of the diameter of the generating circle which is perpendicular to the base OB.

Suggestion. At any instant of the motion of the generating circle, T (its lowest point) is at rest, and the motion of every point of the generating circle is for the moment the same as if it described a circle about T. Hence the normal to the cycloid at P must pass through T.

4. If $\theta = \frac{\pi}{2}$, write the equation of the tangent to the cycloid.

Note. The cycloid was much studied by the most eminent mathematicians of Europe during the first half of the 17th century. In particular Galileo and Pascal discovered many of its properties. Its area was found to be three times the area of the generating circle by Roberval in 1634. The method of drawing tangents (exercise 3) was discovered by Descartes.

131. The Hypocycloid. This locus is the curve traced by a fixed point on the circumference of a circle which rolls internally along the circumference of a fixed circle.

To derive the parametric equations of the hypocycloid, let the radii of the fixed and rolling circles be a and b respectively. Let A be one of the positions in which the tracing point lies on the fixed circle. Take the center of the fixed circle O as origin, and the line OA as X-axis. Let C be the center of the rolling circle when the tracing point has arrived at the position $P(x, y)$, θ the angle through which the line of centers OCB has turned, and ϕ the angle through which the radius CP of the rolling circle has rotated since P left the position A. The coördinates of P are the coördinates of C plus the projections of CP upon the X- and Y-axes, respectively. Hence (Fig. 101), we have

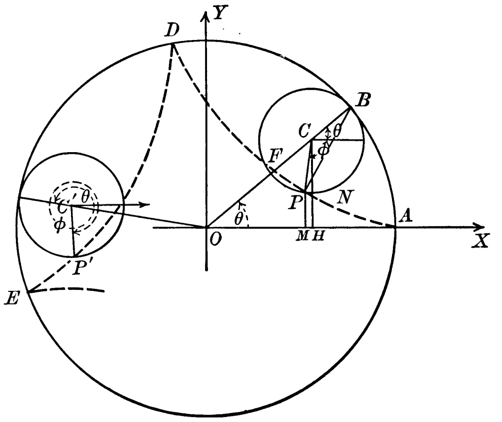

FIG. 101

$$x = OM = OH + NP = OC \cos\theta + CP \cos\phi = (a-b)\cos\theta + b\cos\phi,$$

$$y = MP = HC - NC = (a-b)\sin\theta - b\sin\phi.$$

But arc $PB =$ arc AB, and therefore we have

$$b(\theta + \phi) = a\theta, \text{ or } \phi = \frac{a-b}{b}\theta.$$

Hence, the required equations are

$$x = (a-b)\cos\theta + b\cos\left[\frac{a-b}{b}\theta\right],$$

$$y = (a-b)\sin\theta - b\sin\left[\frac{a-b}{b}\theta\right].$$

132. Special Hypocycloids. If $a = 2b$, the equations of the hypocycloid become

$$x = 2b \cos\theta,$$
$$y = 0.$$

Hence, in this case, the hypocycloid consists of that portion of the X-axis which is included within the fixed circle.

If $a = 4b$, the equations become

$$\left.\begin{aligned} x &= \frac{a}{4}(3\cos\theta + \cos 3\theta),\\ y &= \frac{a}{4}(3\sin\theta - \sin 3\theta). \end{aligned}\right\} \qquad (1)$$

But, from trigonometry,

$$\cos 3\theta = 4\cos^3\theta - 3\cos\theta,$$
$$\sin 3\theta = 3\sin\theta - 4\sin^3\theta.$$

Substituting these values in (1), we get

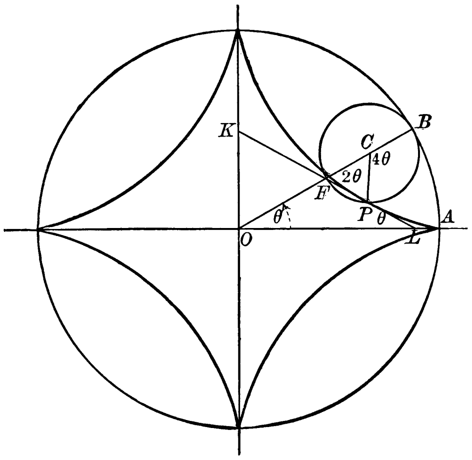

FIG. 102

$$x = a\cos^3\theta,$$
$$y = a\sin^3\theta,$$

from which

$$\cos\theta = \left(\frac{x}{a}\right)^{\frac{1}{3}},$$

$$\sin\theta = \left(\frac{y}{a}\right)^{\frac{1}{3}}.$$

Squaring, adding, and clearing of fractions, we have

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}.$$

The tracing point on the rolling circle reverses its direction of motion at each of the positions in which it is in contact with the fixed circle. These points are called **cusps**. Thus, in Fig. 101, the points A, D, E, are cusps.

If $a = 4b$, there are four cusps, since the rolling circle makes exactly four complete revolutions in returning to its original position. Hence, the locus is called the **four-cusped hypocycloid**. It is an algebraic curve, since its equation, $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$, is algebraic. The equation is readily rationalized, and it then has the form

$$(x^2 + y^2 - a^2)^3 - 27\, a^2x^2y^2 = 0,$$

from which it is seen that the curve is of 6th order. The form of the curve appears in Fig. 102.

EXERCISES

1. Show that the tangent and normal at any point P of an hypocycloid pass through the extremities of that diameter of the rolling circle which passes through the center of the fixed circle (in Fig. 101, the tangent and normal at P pass through F and B respectively).

2. Prove that the length of the tangent at any point of the four-cusped hypocycloid, which is included between the coördinate axes, is equal to the radius of the fixed circle.

Let P (Fig. 102) be any point of the curve, and C, the center of the rolling circle when the tracing point has the position P. The tangent at P is then PF, meeting the axes in K and L. We are to show that KL is equal to the radius of the fixed circle. Since arc $AB =$ arc PB and the radius of the fixed circle is four times the radius of the rolling circle, it follows that the angle BCP is four times the angle BOA; that is, angle $BCP = 4\,\theta$. Hence, angle $BFP = 2\,\theta$; and OFL is an isosceles triangle, $OF = FL$. Also OFK is an isosceles triangle and $OF = FK$. Therefore $LK = 2 \cdot OF = a$.

NOTE. When a straight line, or curve, moves according to a given law, it is generally continuously tangent to another curve called the **envelope**. Thus, the four-cusped hypocycloid is the envelope of a line of constant length which moves so that its extremities are always on the coördinate axes. This property enables one to construct the four-cusped hypocycloid by merely drawing a series of lines of constant length whose extremities all lie on the coördinate axes. The student should make the construction.

133. The Epicycloid. The locus is the curve traced by a fixed point on a circle which rolls externally on the circumference of a fixed circle.

The parametric equations of the epicycloid are found in exactly the same way as were the equations of the hypocycloid (Art. 131). They may be written from the equations of the hypocycloid by changing the sign of b. The equations are

$$x = (a+b)\cos\theta - b\cos\left[\frac{a+b}{b}\theta\right]$$

$$y = (a+b)\sin\theta - b\sin\left[\frac{a+b}{b}\theta\right].$$

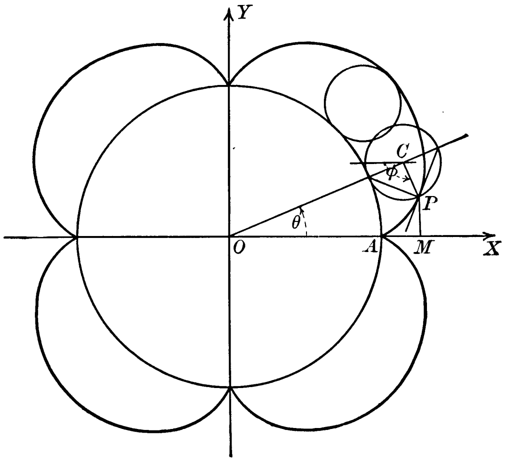

FIG. 103

134. The Cardioid. When the rolling circle is equal to the fixed circle; that is, when $a = b$, the equations of the epicycloid become

$$x = 2a\cos\theta - a\cos 2\theta,$$
$$y = 2a\sin\theta - a\sin 2\theta.$$

To show that this curve is the cardioid (Art. 129, exercise 1), let C be the center of the rolling circle when the tracing point has the position P. Draw PA and let it meet the fixed circle again at Q. Now, since arc $PF =$ arc AF and $FC = FO$, the angle $FCP =$ angle $FOA = \theta$; $FP = FA$ and PQ is parallel to OC. Again, since $OA = OQ$ and angle $OAQ = \theta$, angle $OQA = \theta$ and $QOCP$ is a parallelogram. Therefore, $QP = OC = 2a$, and the point P can be located by laying off the distance $2a$ from Q along the chord of the fixed circle through A. This agrees with the definition of the limaçon for which $b = 2a$. Hence, *the special epicycloid for which the rolling circle is equal to the fixed circle is the* *same as the special limaçon for which the distance laid off along the chords of the fixed circle is twice the radius of the fixed circle.*

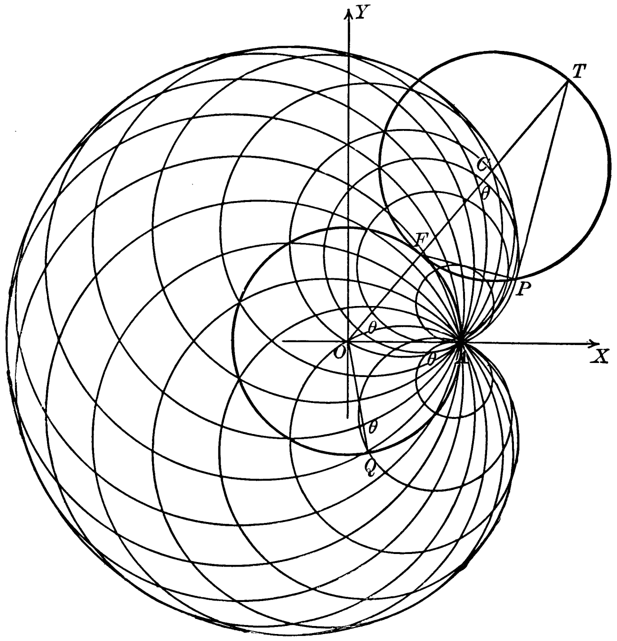

FIG. 104

The tangent to the cardioid at P passes through T, and the normal through F. Hence, the circle whose center is F and whose radius is FP touches the cardioid at P. But this circle passes through A, since $FP = FA$. Therefore, *all the circles having their centers on the fixed circle and passing through A, touch the cardioid. Or, in other words, the cardioid is the envelope of this system of circles.* This property enables one to construct the cardioid by drawing a number of circles of the system.

While the epicycloids are in general transcendental curves, the cardioid, as we have seen, is an algebraic curve.

MISCELLANEOUS EXERCISES

1. A circle of radius a rolls along a fixed straight line; a point on a fixed radius of the circle at a distance b from the center describes a curve called the **trochoid**. Show that the parametric equations of the trochoid are

$$x = a\theta - b \sin \theta,$$
$$y = a - b \cos \theta.$$

Plot the trochoids for which $b < a$ and $b > a$.

2. Show that the polar equation of the cardioid (Fig. 104) is

$$r = 2\,a(1 - \cos \theta),$$

A being the pole and OA the polar axis.

3. Write the parametric equations of the hypocycloid for which $a = 3\,b$. This curve is called the **three-cusped hypocycloid.**

4. A thread is wound around a circular disk and then unwound, kept always stretched. Any point in the thread describes a curve called the **involute of the circle**. If a is the radius of the circle, A the position where the tracing point leaves the circle, O (the center of the circle) the origin, and OA the X-axis, show that the parametric equations of the locus are

$$x = a \cos \theta + a\theta \sin \theta,$$
$$y = a \sin \theta - a\theta \cos \theta.$$

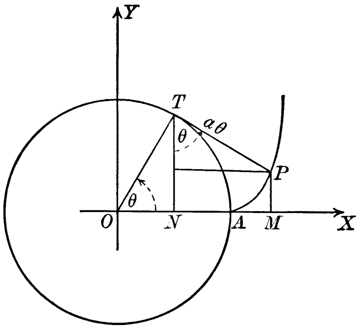

FIG. 105

5. A ladder stands upright against a perpendicular wall and then slides down, the upper end continually resting against the wall. What is the envelope of the moving ladder? What is the locus of its middle point? What are the loci of the points dividing the ladder in the ratio $\frac{m}{n}$?

6. A projectile leaves the muzzle of a gun with a velocity of v feet per second, the barrel of the gun being elevated at an angle ϕ from the horizontal. Neglecting the resistance of the atmosphere, show that the path of the projectile is a parabola whose parametric equations are

$$x = vt \cos \phi,$$
$$y = vt \sin \phi - \frac{gt^2}{2},$$

the parameter t denoting time measured in seconds, and g, the force of gravity.

Take the position of the gun as origin and the horizontal line OL as X-axis. Now, at the end of t seconds, the projectile would be at $Q(\,OQ = vt)$ were it not for gravity which pulls it down a distance $QP = \frac{gt^2}{2}$. Hence,

$$x = OD = vt \cos \phi$$

$$y = DP = DQ - PQ = vt \sin \phi - \frac{gt^2}{2}.$$

Eliminating t from these equations,

$$y = x \tan \phi - \frac{gx^2}{2\,v^2} \cos^2 \phi.$$

Hence, the locus is a parabola.

7. The distance OL (Fig. 106) from the gun to the point where the projectile strikes the horizontal line is called the **range**. Derive a formula for computing the range when the velocity v and the angle of elevation ϕ are given. Show that the greatest range is obtained when $\phi = 45°$. If the velocity v is 1000 ft. per second and the range is 5 miles, what must be the angle of elevation?

8. Two men compete in putting the shot. Compute the effect of any difference in height of the men, other things being equal.

9. A form is to be constructed for a parabolic arch of cement work. The height of the arch is h and the span $2\,l$; find the equation of the arch.

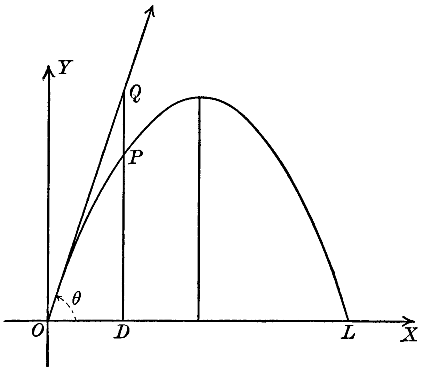

Fig. 106

10. Find the equation of the locus of the foot of the perpendicular from the center upon the tangent to the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

11. The hypotenuse of a right triangle is given in position and length. Find the equation of the locus of the center of the circle inscribed in the triangle.

12. Find the locus of the center of a circle which touches two given circles. Discuss the problem for the various positions which the given circles may have.

13. Through a given point (x_1, y_1) two lines are drawn which meet the coördinate axes in the points A, B and A_1, B_1, respectively. Find the locus of the point of intersection of the lines AB_1 and A_1B.

EMPIRICAL EQUATIONS AND THEIR LOCI

Pairs of corresponding values of two variable quantities are often found by experiment; the graph or locus determined by these pairs (Art. 40) exhibits the change in function due to a change in variable within the limits of observation. It is often of importance to determine the equation of this graph or locus; or, to speak more accurately, to find an equation whose locus coincides as closely as possible with the locus formed from the observed pairs of values. An equation found in this way is called empirical, because it depends upon experiment or observation. An **empirical equation** is the mathematical statement of an empirical law.

The complete solution of the problem of finding an empirical equation from a given set of observations often leads to very intricate analysis, but it is not difficult to test a set of observations to see if it satisfies any one of a number of typical equations. These typical equations embody the simple laws of natural science.

135. Typical Equations.

1. $y = mx + k$; straight lines.
2. $y = Cx^n$; parabolic curves (Fig. 107).
3. $(y - k) = C(x - h)^n$; parabolic curves, origin not at the vertex.
4. $y = \frac{a}{x^n}$; hyperbolic curves (Fig. 107).
5. $(y - k) = \frac{a}{(x - h)^n}$; hyperbolic curves, origin not at the center.
6. $\left.\begin{array}{l} y = ab^x \\ (y - k) = ab^x \end{array}\right\}$; exponential curves (Fig. 109).
7. $y = \frac{a}{b + x^2}$ (Fig. 110).
8. $y = a + bx + cx^2 + dx^3 + \cdots + kx^n$.

136. Loci of typical equations. A study of the foregoing equations and the general characteristics of the corresponding loci is of fundamental importance.

Figure 107 shows a number of curves belonging to the family $y = Cx^n$. This is an especially important family of curves, since many of the simple laws of natural science are embodied in the equation $y = Cx^n$. The curves are drawn for $C = 1$; the general form of the curves will be the same for any other value of C.

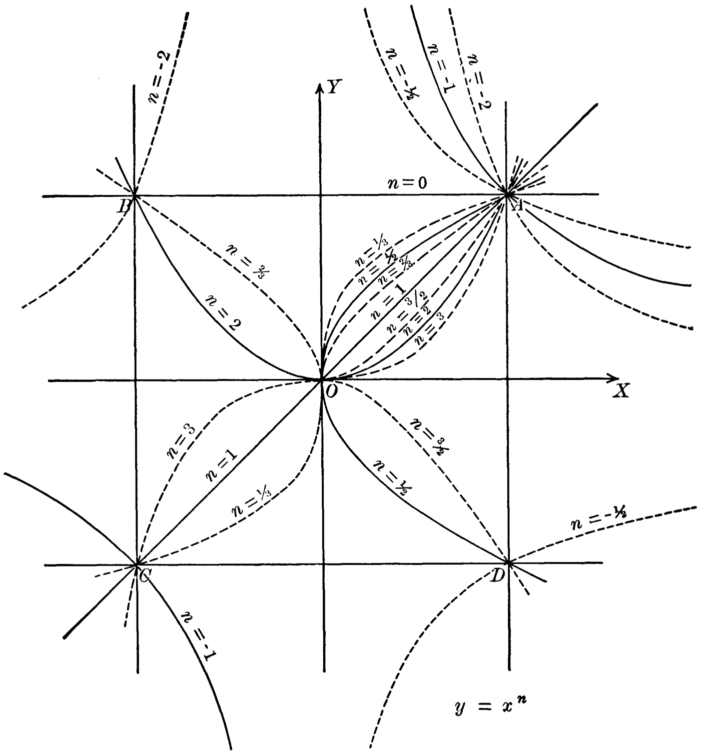

FIG. 107

If n is a positive number (type 2, Art. 135), the curves are called **parabolic.** If n is 3 or $\frac{1}{3}$, the curve is the **cubical parabola.** If n is $\frac{3}{2}$ or $\frac{2}{3}$, the curve is the **semicubical parabola.** If n is 2 or $\frac{1}{2}$, the curve is the *ordinary parabola.* If n is a negative number, the curves are called **hyperbolic.** If n is -1, the curve is the *ordinary hyperbola.*

A study of the figure reveals the following properties:

I. *All curves of the system, whatever the value of n, pass through the point* $A \equiv (1, 1)$.

II. *All the parabolic curves of the system* $(n > 0)$ *pass through the origin, follow the diagonals of the square ABCD more or less closely, and pass out of the square through A and one other vertex.*

III. *If n is a positive number represented by* $\frac{a}{b}$ *(a and b prime to each other), then*

(1) *when a is even and b is odd, the curves pass through*

$$B \equiv (-1, 1);$$

(2) *when a is odd and b is odd, the curves pass through*

$$C \equiv (-1, -1); \text{ and}$$

(3) *when a is odd and b is even, the curves pass through*

$$D \equiv (1, -1).$$

IV. *The parabolic curves of the system fill the square ABCD and the infinite regions of the plane which corner on this square; i.e. the shaded regions in Fig.* 108.

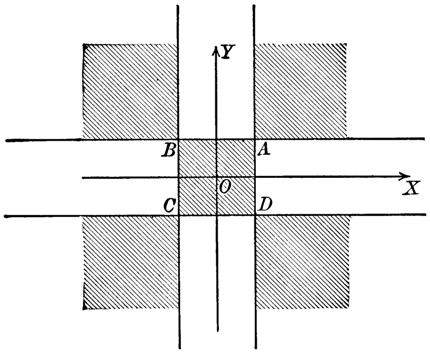

FIG. 108

V. *When n is a positive even integer, the curves touch the X-axis more and more closely the larger n is taken; i.e. the curvature at the origin becomes less and less as the value of n is increased.*

When n is a positive odd integer, the curves touch the X-axis at the origin, but the curvature changes from concave downward on the left of the Y-axis to concave upward on the right. Each curve has a point of inflexion at the origin.

When n is fractional, with neither numerator nor denominator equal to unity, each curve has a cusp at the origin.

VI. *The hyperbolic curves of the system* ($n < 0$) *fill the regions of the plane outside the square ABCD and the infinite regions cornering on this square ; i.e. the unshaded regions in Fig.* 108. *The axes are asymptotes to each hyperbola of the system.*

Types 3 and 5 (Art. 135) do not differ in form from types 2 and 4.

Type 6 is illustrated in Fig. 109. Each curve of the system passes through the point (0, 1) (a is assumed to be unity in drawing these curves), the curvature depending upon the value of b. The curves illustrate phenomena that follow the "compound interest law."

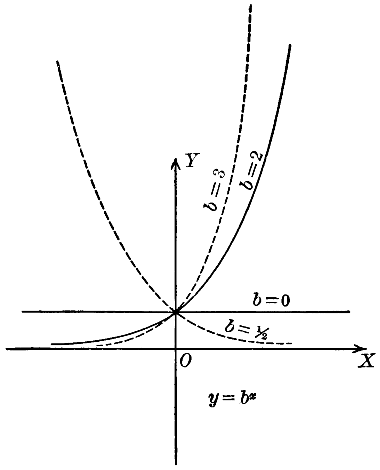

FIG. 109

Type 7 is shown in Fig. 110. If $a^2 = b^3$, the curve is the witch (Fig. 98). This curve is of special importance in representing phenomena where the observed value (the function) gradually decreases, from a maximum

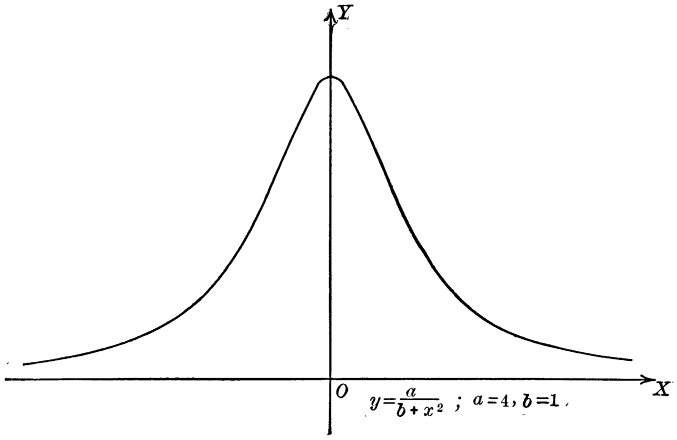

FIG. 110

at $x = 0$, as the variable increases in value. Other curves applicable to phenomena of this character are the probability curve,

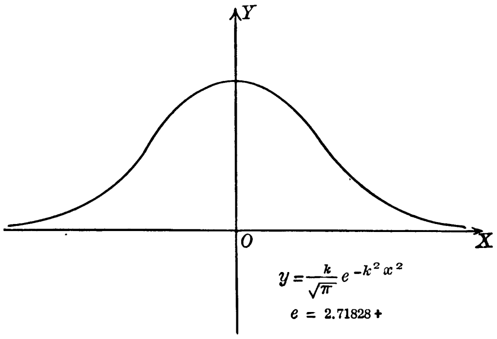

FIG. 111

$y = \frac{k}{\sqrt{\pi}} e^{-k^2x^2}$, Fig. 111, and the curve $y = \frac{2a}{e^x + e^{-x}}$, Fig. 112. If $a = 1$, the latter is the hyperbolic secant curve.

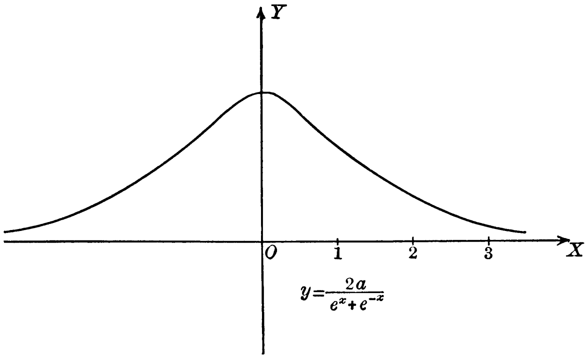

FIG. 112

137. Selection of type curve and determination of constants. Frequently the law which experimental data must follow is known beforehand, and then it is only necessary to determine the constants in the equation. For example, in experiments on falling bodies, the law is known to be of the form $y = Cx^2$, where y represents the distance fallen during the time x. In this case, one pair of values of x and y will determine the value of the constant C.

Experimentally determined values of any function are never absolutely exact, so that plotted points, determined from experiment, never all lie exactly on the curve representing the known law. The values of the constants, determined as above, are therefore more or less approximate. The aim is to find such values for the constants as will give the best average curve to represent the observed values of the function.

In case the law is not known, the curve which best represents the observed values of the function must be selected by trial. The procedure is as follows:

(*a*) Plot the observed values carefully;

(*b*) From the known forms of curves discussed in Art. 136, or elsewhere, select that one which resembles the plotted curve most closely;

(*c*) Determine the constants in the equation of the selected curve so that it will fit the observed values most closely.

To fulfill the requirements (*b*) and (*c*) satisfactorily requires good judgment and a good eye as well as some knowledge of the forms of various types of curves. The results obtained are often quite as serviceable as though more intricate analysis had been employed to find them.

138. Test by means of linear equations. After a trial curve has been selected, it is often rather difficult to determine whether this curve actually represents the observed values of the function with sufficient accuracy for the purposes of the problem or not. The following device is of great assistance in determining whether to retain or reject the trial curve. The typical equations in Art. 136 can be transformed into linear equations as follows:

2. $y = Cx^n$ can be written $\log y = \log C + n \log x$.

3. $(y - k) = C(x - h)^n$ can be written $\log (y - k) = \log C + n \log (x - h)$.

Types 4 and 5 are included in the above.

6. $y = ab^x$ can be written $\log y = \log a + x \log b$.

7. $y = \dfrac{a}{b + x^2}$, can be written $\dfrac{1}{y} = \dfrac{b}{a} + \dfrac{x^2}{a}$.

If the observed values satisfy sufficiently accurately an equation of the form $y = Cx^n$, say, then the logarithms of the observed values must satisfy the equation $\log y = \log C + n \log x$. Hence, the points plotted from the logarithms of the observed values must lie closely upon a straight line. If they do not lie upon a straight line, or nearly so, the required equation is not of the form $y = Cx^n$.

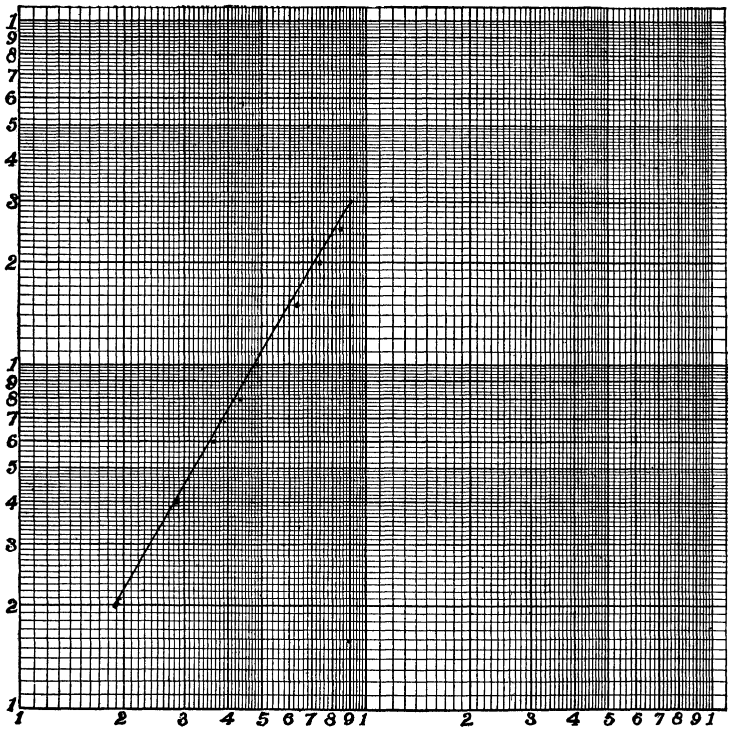

Fig. 113

Similarly, if the observed values are to satisfy an equation of the type $y = ab^x$, then the values of $\log y$ and x must satisfy the equation $\log y = \log a + x \log b$.

Again, if the observed values are to satisfy an equation of the type 7, the values of $\frac{1}{y}$ and x^2 must satisfy the equation

$$\frac{1}{y}=\frac{b}{a}+\frac{x^2}{a}.$$

Instead of looking up the logarithms of the numbers in a given table, logarithmic paper may be used. The horizontal and vertical scales on this paper represent the logarithms of numbers. Figure 113 shows a sheet of this paper on which has been plotted the table in Example II, Art. 139. Because the plotted points lie very closely upon a straight line, we may assume the equation $y = Cx^n$.

139. Examples and exercises. The foregoing statements will be better understood from the following illustrative examples and exercises.

Example I. Find an equation which is satisfied by the following pairs of values of x and y:

$x = 1$	2	3	4	5	6
$y = 3$	1.5	1.22	1.125	1.08	1.06

Plotting the given pairs of values, we have the curve in Fig. 114. We now note that this curve resembles one of the hyperbolic curves belonging to the system $y = Cx^n$, but with this difference; instead of approaching the X-axis as an asymptote, it apparently approaches the line $y = 1$ as an asymptote. We therefore assume, as a trial equation, $y - 1 = Cx^n$. If the given

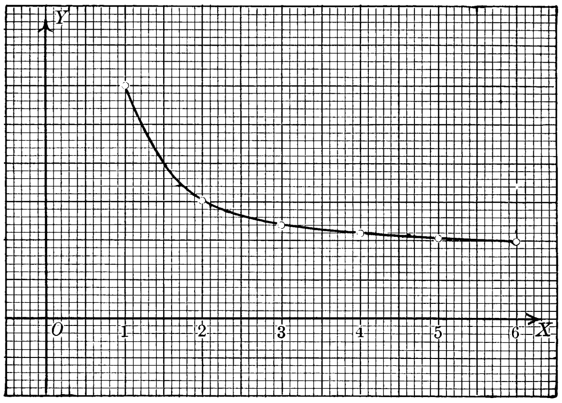

FIG. 114

values of x and y satisfy this equation, then the values of $\log(y-1)$ and $\log x$ must satisfy the equation

$$\log(y-1) = \log C + n \log x,$$

or the equation of a straight line. From the given values of x and y, we obtain,

$\log x =$	0	.301	.477	.602	.70	.80
$\log(y-1) =$	.301	$-.301$	$-.658$	$-.903$	-1.10	-1.22

Plotting these values, we obtain Fig. 115. We see that the straight line passing through the first and next the last of these points very nearly passes through the others. The slope of this line is -2 and the intercept on the Y-axis is $.301 = \log 2$. Hence, $\log(y-1) = \log 2 - 2 \log x$, or

$$y = 1 + \frac{2}{x^2}.$$

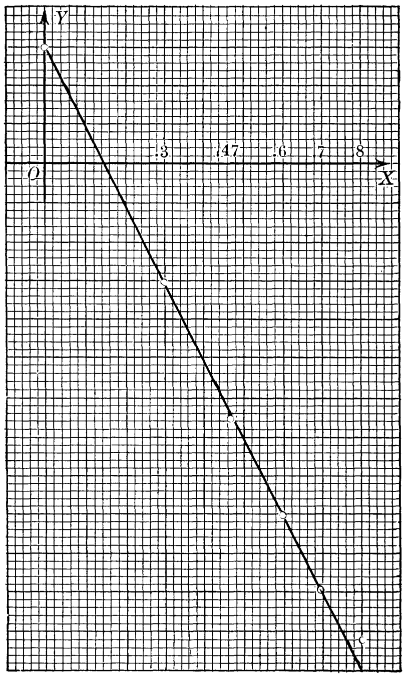

FIG. 115

By actual substitution, this equation is seen to be very closely satisfied by the given pairs of values of x and y.

Logarithmic paper may be used in this example, as explained in the preceding article.

Example II. For water flowing in pipes, the loss of pressure due to friction is approximately proportional to the square of the velocity. If y is the loss of pressure per 1000 feet and x is the velocity in feet per second, then (approximately) $y = ax^2$. Find the value of the constant a so that the following experimental data will fit the given equation closely:

$x =$	1.9	2.8	3.6	4.3	4.8	6.1	7.2	8.2	9.1
$y =$	2	4	6	8	10	15	20	25	30

Example III. Show that the observed data in the preceding example will more closely satisfy an equation of the type $y = ax^n$. See Fig. 113.

Example IV. The following observations were made of wind pressure on inclined surfaces.

Inclination from vertical: 30°, 40°, 50°, 60°, 70°.
Pressure (pounds per square foot): 5.5, 5.3, 4.4, 3.5, 2.1.
Determine the curve representing the pressure as a function of the angle.

Suggestion. Assume the equation $k - y = ax^n$. Plot the observed values. Estimate $k = 6$. Plot the values of the logarithms of $6 - y$ and x, and fit a straight line to the plotted points.

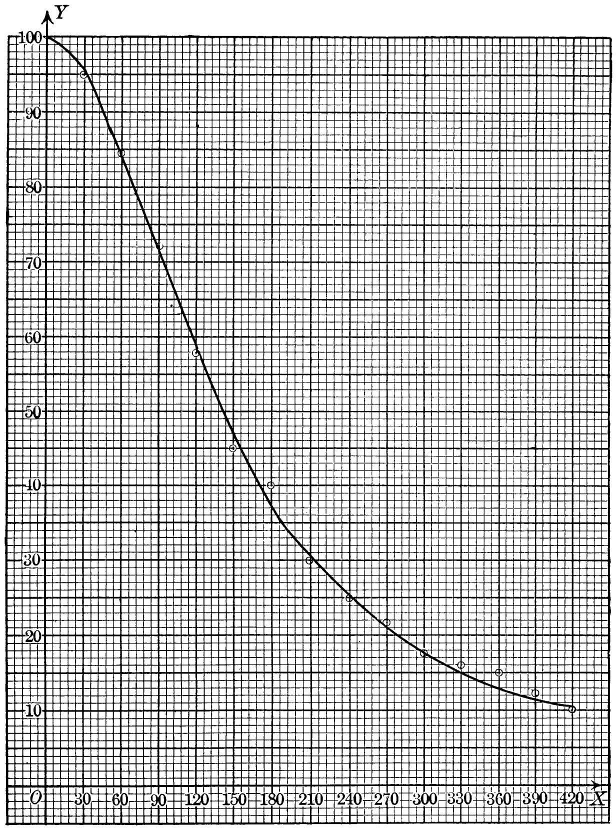

Fig. 116

Example V. In a certain investigation upon the strains in railway bridges due to the passage of trains, the following data were found:

$x =$	0	30	60	90	120	150	180	210	240	270	300	330	360	390	420
$y =$	100	95	84	72	58	45	40	30	25	22	18	16	15	12	10

Find an empirical equation which these observations will satisfy with close approximation.

SUGGESTION. Plot the given pairs of values carefully. Note that the curve obtained seems to approach the X-axis as an asymptote. Since the function begins with a maximum value at $x = 0$ and steadily decreases in value as x increases, choose type 7 as a trial equation. Plot the values of $\frac{1}{y}$ and x^2 and fit a straight line to the plotted points.

Figure 116 shows the points plotted from the given pairs of values of x and y and also the locus of the equation $y = \frac{a}{b + x^2}$ in which $a = 2{,}000{,}000$ and $b = 20{,}000$. The scales are indicated on the figure.

Example VI. In a series of experiments on the adiabatic expansion for air, the following data were obtained, where v stands for volume and p for the corresponding pressure.

$v =$	3	4	5.2	6.0	7.3	8.5	10.0
$p =$	107.3	71.5	49.5	40.5	30.8	24.9	19.8

Find the empirical equation connecting p and v.

SUGGESTION. Since the curve obtained from the given pairs of values of p and v resembles one of the hyperbolas of the system $y = Cx^n$, plot the values of $\log p$ and $\log v$ and fit a straight line to the plotted points. The equation sought is $pv^{1.4} = 497.7$.

EXERCISES

1. If l represents the length of a steel bar and t represents temperature, find the equation connecting l and t from the following observations:

$l =$	1	1.0004	1.0008	1.0012	1.0016	1.0024	1.0040
$t =$	0	20	40	60	80	120	200

2. Find the equation connecting Q and h from the following set of observations:

$h =$	.583	.667	.750	.834	.876	.958
$Q =$	7.000	7.600	7.940	8.420	8.680	9.040

3. Show that the following set of corresponding values satisfies an equation of the form $y = ab^x$. Find the values of a and b.

$x =$	2.000	3.20	4.70	8.5	10.3	12.6
$y =$	7.086	12.64	125.07	163.0	388.4	1178.0

4. The following set of observations represents the deflection d of a beam of length L. Find the equation connecting d and L.

$L =$	12	16	20	24	28	32	36	40
$d =$	.17	.043	.085	.145	.220	.342	.512	.713

5. Find the equation connecting u and v from the following set of corresponding values:

$u =$	.5	1.1	1.70	2.30	5.10	6.40
$v =$	13.6	4.0	2.37	1.84	1.33	1.28

140. Type $y = a + bx + cx^2 + dx^3 + \cdots + kx^n$. When a given set of corresponding values will not satisfy, in a satisfactory manner, any of the type-equations 1 to 7 (Art. 137), the general equation

$$y = a + bx + cx^2 + dx^3 + \cdots + kx^n$$

may be assumed. By substituting pairs of corresponding values in this equation, the values of the constants $a, b, c, \cdots k$ can be determined and may often be so adjusted that the locus of the resulting equation will represent the function to a fair degree of approximation within the limits of observation.

EXERCISES

1. Show that the following set of corresponding values satisfy an equation of the form $y = a + bx + cx^2$. Find the values of a, b, c:

$x =$	8	23	39	53	63
$y =$	10	19	27	33	36

2. Find an equation of the form $y = a + bx + cx^2$ which will be satisfied by the corresponding values of angle and wind pressure in Example IV, Art. 139. Why is the equation found in this way not as satisfactory as the equation found in Example IV?

PART II

SOLID ANALYTIC GEOMETRY

CHAPTER X

SYSTEMS OF COÖRDINATES

141. Rectangular and oblique coördinates. As has been said, it requires one number to locate a point on a line and two numbers to locate a point in a plane (Art. 4). To locate a point in space it requires three numbers, called the **coördinates** of the point. These coördinates may be chosen in several different ways; any particular way of choosing them gives rise to a **system of coördinates.** Thus (Fig. 117), let OX, OY, and OZ be three linear scales having a common origin O and not lying in the same plane. They determine in pairs three planes XOY, YOZ, and XOZ, called the **coördinate planes.** If through any point in space, as P, three planes are drawn parallel to the coördinate planes, they intersect the linear scales in the points D, E, and F. The distances $x = OD$, $y = OE$, and $z = OF$ are the **Cartesian coördinates** of the point P. The linear scales OX, OY, and OZ are called the X-, Y-, and Z-axes, respectively. The coördinate planes XOY, XOZ, and YOZ are called the XY-, XZ-, and YZ-planes, respectively. The sys-

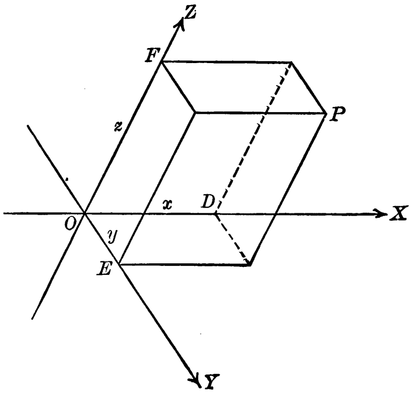

FIG. 117

tem of coördinates thus set up is called the **Cartesian system of coördinates.**

When the axes, OX, OY, and OZ are mutually perpendicular, the system of coördinates is called **rectangular or orthogonal.** If the axes are not mutually perpendicular, the system is called **oblique.** From the definition of the coördinates of a point, and the definition of a linear scale, it follows that, in the Cartesian system of coördinates, to each point in space there corresponds one set of values of x, y, z; and to each set of values of x, y, z there corresponds one point in space.

EXERCISES

(In the following exercises, take the axes to be mutually perpendicular. Cross-section paper may be used.)

1. Plot to scale the following points, the coördinates being always written in the order (x, y, z):

$(1, 1, 1)$, $(2, 0, 3)$, $(-4, -1, -3)$, $(-4, 2, 8)$, $(0, 0, 2)$, $(1, -3, 0)$.

2. Find the distance between the points $(1, -2, 3)$ and $(-1, 2, -2)$.

3. Where are the points located for which $x = 0$? $y = 0$? $z = 0$? What are the equations of the coördinate planes? Where are the points located for which $x = a$; $y = b$; $z = c$? What are the equations of the planes parallel to the coördinate planes?

4. Where are the points located for which $x = 0$ and $y = 0$? for which $x = a$ and $y = b$? for which $x = y$? for which $x = y = z$?

5. The points $(2, 2, 3)$, $(2, 4, 3)$, $(4, 2, 3)$, and $(3, 3, 2)$ are four of the vertices of a parallelopipedon. Find the coördinates of the remaining four vertices. Is there more than one solution to this problem?

142. Spherical coördinates. Let OX, OY, OZ (Fig. 118) be a set of rectangular axes, and P any point in space. The distance $OP = r$, the angle $ZOP = \theta$, and the angle which the plane ZOP makes with the fixed plane $XOZ = \phi$ are the **spherical coördinates** of the point P. They are written in the order (r, θ, ϕ).

If the point P is on the surface of the earth, then θ is the *co-latitude* and ϕ is the *longitude* of P. If P is on the celestial sphere, then θ is the *co-declination* and ϕ the *right ascension* of P. If Z is the zenith, then θ is the *zenith distance* and ϕ is the *azimuth* of P.

143. Cylindrical coördinates. In Fig. 118, let $OD = r'$, the angle $XOD = \phi$, and $DP = z$; (r', ϕ, z) are the **cylindrical coördinates** of P. Again, let α, β, γ denote the angles which $OP = r$ makes with the X-, Y-, and Z-axes, respectively; then $(r, \alpha, \beta, \gamma)$ are the **polar coördinates** of P.

Spherical coördinates and cylindrical coördinates are modifications of polar coördinates in space. Each is in common use and each has its advantages. Spherical coördinates are especially useful in astronomy and in geodetic surveying.

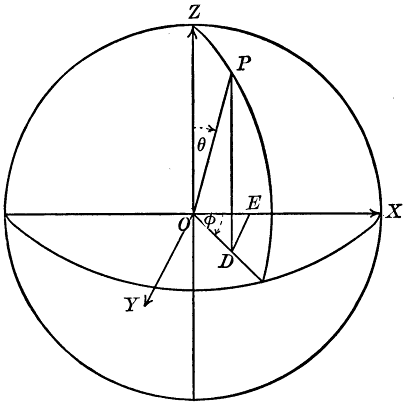

Fig. 118

EXERCISES

1. Using Fig. 118, show that the rectangular coördinates of P and the spherical coördinates of P are connected by the following formulas:

$$x = r \sin\theta \cos\phi,$$

$$y = r \sin\theta \sin\phi,$$

$$z = r \cos\theta.$$

Conversely, show that

$$r^2 = x^2 + y^2 + z^2,$$

$$\tan^2\theta = \frac{x^2 + y^2}{z^2},$$

$$\tan\phi = \frac{y}{x}.$$

2. What are the formulas connecting the rectangular coördinates of P with the cylindrical coördinates of P?

3. Will a given set of integral or fractional values of r, θ, ϕ or of r', ϕ, z locate one and only one point in space? Does a given point in space have more than one set of polar coördinates?

4. Locate the points whose spherical coördinates are: $(3, 30°, 60°)$, $\left(2, \frac{\pi}{4}, \pi\right)$, $(1, 45°, 45°)$. Find the rectangular coördinates of these points.

5. Find the spherical coördinates and also the cylindrical coördinates of the following points: $(2, 3, 4)$, $(3, 3, -2)$, $(-1, -2, 1)$.

6. Where are the points located for which $r = \text{const.}$? for which $\theta = \text{const.}$? for which $\phi = \text{const.}$? for which $r' = \text{const.}$?

7. Where are the points located for which $\theta = \text{const.}$ and $\phi = \text{const.}$? for which $\phi = \text{const.}$ and $r = \text{const.}$? for which $r = \text{const.}$ and $\theta = \text{const.}$? for which $r' = \text{const.}$ and $z = \text{const.}$?

CHAPTER XI

DIRECTED SEGMENTS IN SPACE

144. Projections upon the coördinate axes. As in plane geometry, we shall call a segment of a straight line to which a direction has been attached, a **directed line-segment**, or simply, a **directed segment**. If P_1P_2 is a directed segment, then P_1 is called the **initial point**, and P_2, the **terminal point.**

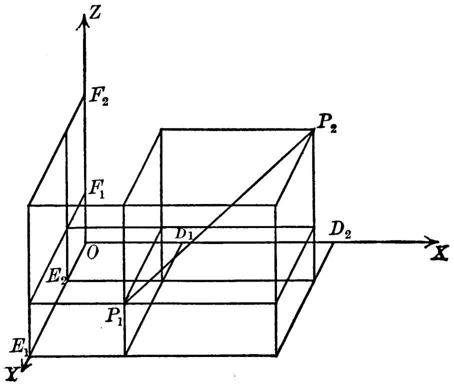

Fig. 119

If planes are drawn through the initial and terminal points of a directed segment and perpendicular to each of the coördinate axes in turn, these planes will determine upon each axis a segment called the **projection** of the given directed segment upon that axis. In Fig. 119, let $P_1 \equiv (x_1, y_1, z_1)$ and $P_2 \equiv (x_2, y_2, z_2)$; then we have

$$\text{projection of } P_1P_2 \text{ upon the } X\text{-axis} = x_2 - x_1.$$
$$\text{projection of } P_1P_2 \text{ upon the } Y\text{-axis} = y_2 - y_1.$$
$$\text{projection of } P_1P_2 \text{ upon the } Z\text{-axis} = z_2 - z_1.$$

145. Length of segment. A segment P_1P_2 is the diagonal of a rectangular parallelopiped whose edges are the projections of the segment upon the coördinate axes. Hence, we have

$$P_1P_2 = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

EXERCISES

1. Find the lengths of the following segments and their projections upon the coördinate axes.

(*a*) $(1, 2, 3), (-2, 1, 1)$; (*b*) $(0, 0, 0), (2, 0, 1)$; (*c*) $(3, -2, 0), (2, 3, 1)$; (*d*) $(0, 4, 1), (-2, -1, -2)$; (*e*) $(0, 3, 0), (3, -1, 0)$.

2. A straight line five units in length has one extremity at the origin and is equally inclined to the coördinate axes. Find its projections upon the axes.

3. The initial point of a directed segment is at the point $(3, 2, -1)$ and its projections upon the X-, Y-, and Z-axes are respectively 4, -6, and -2. Find the coördinates of the terminal point and construct the figure.

4. If the terminal point of a directed segment is $(-1, 3, 5)$ and its projections upon the X-, Y-, and Z-axes are respectively -2, 3, and -6, what are the coördinates of the initial point and the length of the directed segment?

146. Direction angles and direction cosines of a directed segment. The angles which a directed segment makes with the positive directions of the coördinate axes are called the **direction angles** of the segment. The cosines of the direction angles are called the **direction cosines** of the segment.

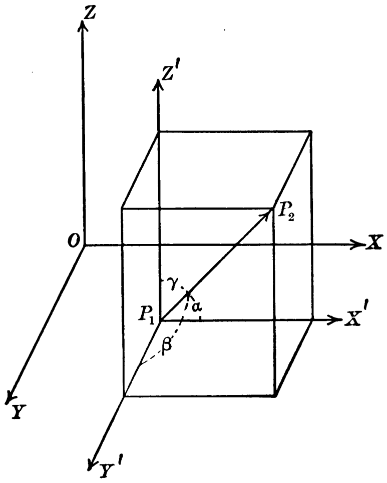

FIG. 120

Through P_1 draw lines parallel to the axes; *i.e.* the lines P_1X', P_1Y', P_1Z' (Fig. 120). The direction angles of P_1P_2 are then,

$$\alpha = X'P_1P_2,$$
$$\beta = Y'P_1P_2,$$
$$\gamma = Z'P_1P_2.$$

If l is the length of P_1P_2, then the direction cosines are given by the equations:

$$\cos \alpha = \frac{x_2 - x_1}{l},$$

$$\cos \beta = \frac{y_2 - y_1}{l},$$

$$\cos \gamma = \frac{z_2 - z_1}{l}.$$

147. Relation connecting the direction cosines of a segment.

THEOREM. *The sum of the squares of the direction cosines of any segment is equal to unity.*

For let l be the length of any segment. Then (Art. 145), we have

$$l^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2.$$

Dividing by l^2, we obtain

$$1 = \left(\frac{x_2 - x_1}{l}\right)^2 + \left(\frac{y_2 - y_1}{l}\right)^2 + \left(\frac{z_2 - z_1}{l}\right)^2,$$

or

$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1.$$

EXERCISES

1. Find the length and the direction cosines of each of the following segments:

$P_1 \equiv (4, 3, -2)$, $P_2 \equiv (-2, 1, -5)$; $P_1 \equiv (4, 7, -2)$, $P_2 \equiv (3, 5, -4)$; $P_1 \equiv (3, -8, 6)$, $P_2 \equiv (6, -4, 6)$.

2. Find the lengths and the direction cosines of each side of the triangle whose vertices are the points $(3, 2, 0)$, $(-2, 5, 7)$, and $(1, -3, -5)$, the sides being taken in the order given.

3. Given the direction cosines of the segment P_1P_2; what are the direction cosines of the segment P_2P_1? What is the direction of a segment when $\cos \alpha = 0$? when $\cos \beta = 0$? when $\cos \gamma = 0$? when $\cos \alpha = \cos \beta = 0$? when $\cos \alpha = \cos \gamma = 0$? when $\cos \beta = \cos \gamma = 0$?

4. A segment is five units long and its initial point is $(-2, 1, -3)$. If $\cos \alpha = \frac{1}{2}$ and $\cos \beta = \frac{1}{3}$, find the coördinates of the terminal point and the projections upon the axes. There are two solutions, find each of them and construct the figure.

5. Show that the direction cosines of each of the lines joining the points $(4, -8, 6)$ and $(-2, 4, -3)$ to the point $(12, -24, 18)$ are the same. How are the points situated?

6. Find the direction angles of the segment drawn from the origin to the point $(8, 6, 0)$. From the origin to the point $(2, -1, -2)$.

7. Show by means of direction cosines that the three points $(3, -2, 7)$, $(6, 4, -2)$, and $(5, 2, 1)$ lie on a straight line.

8. If two of the direction angles of a segment are $\frac{\pi}{3}$ and $\frac{\pi}{4}$, what is the third?

9. Show that the numbers 3, -4, and -2 are proportional to the direction cosines of the segment joining the origin to the point $(3, -4, -2)$.

10. Show that any three real numbers a, b, and c are proportional to the direction cosines of the segment joining the origin to the point (a, b, c).

148. Projection of a segment upon any line. Let P_1P_2 be any segment, and AB any line in space. Through the extremities of the segment draw planes perpendicular to AB. These planes determine a segment CD upon AB which is called the **projection** of P_1P_2 upon AB.

Through P_1 draw a line parallel to AB, meeting the planes in the points P_1 and Q. Let θ represent the angle P_2P_1Q; then

$$CD = P_1Q = P_1P_2 \cos\theta.$$

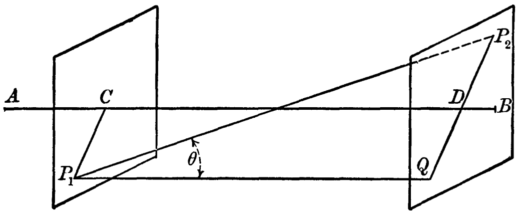

FIG. 121

149. Projection of a broken line. A series of segments so arranged that the terminal point of each is the initial point of the next following and the terminal point of the last is the initial point of the first, constitutes a **closed line**, or polygon, in space. The sum of the projections of the sides of a closed line upon any line in space is clearly equal to zero. It follows from the foregoing property that:

THEOREM. *The sum of the projections of a series of segments joining the point A to the point B, upon any straight line in space, is equal to the projection of the segment AB upon that line.*

For, the succession of segments AP_1, P_1P_2, $P_2P_3 \cdots BA$ forms a closed line, and hence the sum of the projections of its sides upon any line is equal to zero; *i.e.* the sum of the projections of the sides of the broken line joining A to B is equal to the projection of the straight line joining A to B.

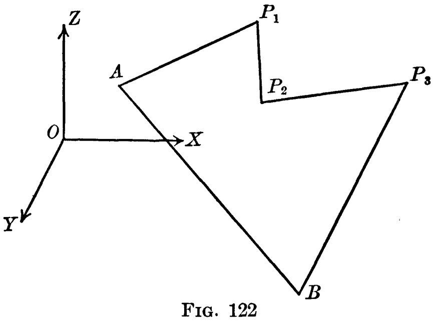

FIG. 122

150. The angle between two segments. When two segments do not intersect, the angle between them is defined to be the angle between two intersecting segments drawn parallel to, and agreeing in direction with, the given segments.

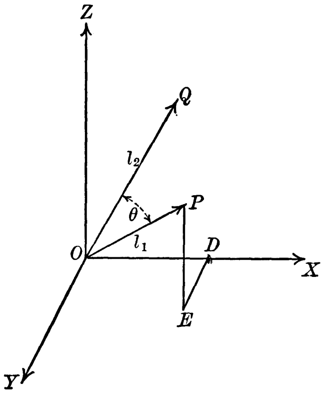

FIG. 123

To find the angle between two given segments in terms of their direction angles, let l_1 and l_2 be the lengths, and $\alpha_1, \beta_1, \gamma_1$; $\alpha_2, \beta_2, \gamma_2$, their respective direction angles. From the origin draw two segments, OP and OQ, having lengths and direction angles equal respectively to the lengths and direction angles of the given segments (Fig. 123). By definition, $\theta = POQ$ is the angle to be found. Let the coördinates of P be $x = OD$, $y = DE$, and $z = EP$. Now, by the preceding article, the projection of the broken line $ODEP$ upon OQ is equal to the projection of OP upon OQ. That is,

$$l_1 \cos\theta = x \cos\alpha_2 + y \cos\beta_2 + z \cos\gamma_2.$$

Dividing through by l_1 and remembering that $\frac{x}{l_1} = \cos\alpha_1$, etc., we have

$$\boldsymbol{\cos\theta = \cos\alpha_1 \cos\alpha_2 + \cos\beta_1 \cos\beta_2 + \cos\gamma_1 \cos\gamma_2.}$$

We will assume that the angle between the given segments is the smallest positive angle satisfying this equation; that is

$$0 \leqq \theta \leqq \pi.$$

151. Perpendicular segments. Parallel segments.

(*a*) *Two segments are perpendicular to each other if*

$$\cos\alpha_1 \cos\alpha_2 + \cos\beta_1 \cos\beta_2 + \cos\gamma_1 \cos\gamma_2 = 0.$$

For then $\cos\theta = 0$ and therefore $\theta = 90°$.

(*b*) *Two segments are parallel and extend in the same direction if their direction angles are equal, each to each.*

For then the expression $\cos\alpha_1\cos\alpha_2+\cos\beta_1\cos\beta_2+\cos\gamma_1\cos\gamma_2$ becomes $\cos^2\alpha_1+\cos^2\beta_1+\cos^2\gamma_1=1$ (Art. 147). Therefore, in this case, $\cos\theta=1$, and $\theta=0°$.

(*c*) *Two segments are parallel and in opposite directions if their angles differ by* 180°, *each from each.*

For then

$$\cos\alpha_1=-\cos\alpha_2,$$
$$\cos\beta_1=-\cos\beta_2,$$
$$\cos\gamma_1=-\cos\gamma_2.$$

Hence the expression $\cos\alpha_1\cos\alpha_2+\cos\beta_1\cos\beta_2+\cos\gamma_1\cos\gamma_2$ becomes $-(\cos^2\alpha_1+\cos^2\beta_1+\cos^2\gamma_1)=-1$. Therefore $\cos\theta=-1$, and $\theta=180°$.

EXERCISES

1. Find the angle between two segments whose direction cosines are as follows:

(*a*) $\frac{6}{7}$, $\frac{3}{7}$, $-\frac{2}{7}$ and $\frac{3}{7}$, $-\frac{2}{7}$, $\frac{6}{7}$; (*b*) $\frac{2}{3}$, $-\frac{2}{3}$, $\frac{1}{3}$ and $\frac{3}{7}$, $\frac{6}{7}$, $\frac{2}{7}$; (*c*) $\frac{2}{3}$, $-\frac{1}{3}$, $\frac{2}{3}$ and $-\frac{3}{13}$, $\frac{4}{13}$, $\frac{12}{13}$.

2. Show that the lines whose direction cosines are $\frac{3}{7}$, $\frac{6}{7}$, $\frac{2}{7}$; $-\frac{2}{7}$, $\frac{8}{7}$, $-\frac{6}{7}$; and $-\frac{6}{7}$, $\frac{2}{7}$, $\frac{3}{7}$ are mutually perpendicular.

3. Show that the points having the coördinates $(-6, 3, 2)$, $(3, -2, 4)$, $(5, 7, 3)$, and $(-13, 17, -1)$ are the vertices of a trapezoid.

4. Show that the points $(7, 3, 4)$, $(1, 0, 6)$, and $(4, 5, -2)$ are the vertices of a right triangle.

5. Show that the points $(7, 2, 4)$, $(4, -4, 2)$, $(9, -1, 10)$, and $(6, -7, 8)$ are the vertices of a square.

6. Prove that if the direction angles of two segments are supplementary, each to each, the segments are parallel and in opposite directions.

7. Find the length of the projection of the segment $P_1\equiv(3, 2, -6)$, $P_2\equiv(-3, 5, -4)$ upon the line drawn from $(1, 2, 3)$ to $(3, 3, 1)$.

8. Find the length of the projection of the segment $P_1\equiv(6, 3, 2)$, $P_2\equiv(4, 2, 0)$ upon the line drawn from $(7, -6, 0)$ to $(-5, -2, 3)$.

152. Point dividing a given segment in a given ratio. Let P be a point on the segment P_1P_2 situated so that $\frac{P_1P}{PP_2}=r$, a given number. Then, by composition, $\frac{P_1P}{P_1P_2}=\frac{r}{r+1}$. Through P draw planes perpendicular to the coördinate axes. These planes divide

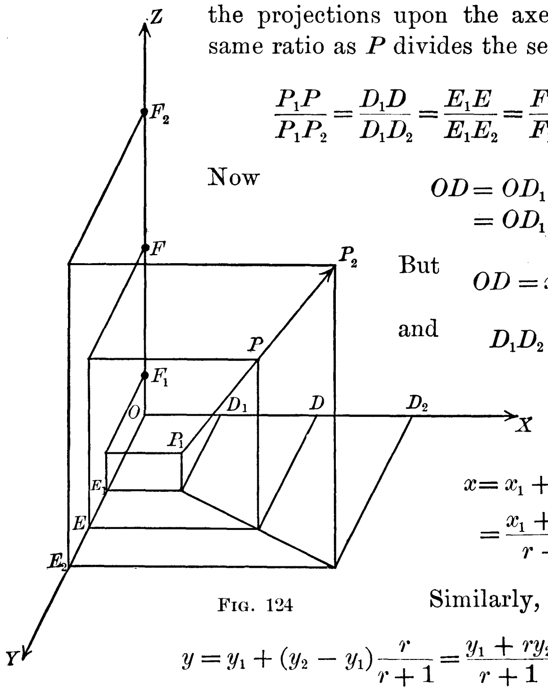

FIG. 124

the projections upon the axes in exactly the same ratio as P divides the segment; that is

$$\frac{P_1P}{P_1P_2}=\frac{D_1D}{D_1D_2}=\frac{E_1E}{E_1E_2}=\frac{F_1F}{F_1F_2}=\frac{r}{r+1}.$$

Now

$$\begin{aligned} OD &= OD_1 + D_1D \\ &= OD_1 + D_1D_2\frac{r}{r+1}. \end{aligned}$$

But $OD = x,\ OD_1 = x_1,$

and $D_1D_2 = x_2 - x_1.$

Substituting, we have

$$\begin{aligned} x &= x_1 + (x_2 - x_1)\frac{r}{r+1} \\ &= \frac{x_1 + rx_2}{r+1}. \end{aligned}$$

Similarly, we obtain

$$y = y_1 + (y_2 - y_1)\frac{r}{r+1} = \frac{y_1 + ry_2}{r+1},$$

$$z = z_1 + (z_2 - z_1)\frac{r}{r+1} = \frac{z_1 + rz_2}{r+1}.$$

EXERCISES

1. Find the coördinates of the point dividing the segment joining the following points in the given ratio r.

(a) $(3, 4, 2), (7, -6, 4), r = 2$. (b) $(7, 3, 9), (2, 1, 2), r = 4$.

2. Show that the coördinates of the point bisecting the segment

$(x_1, y_1, z_1), (x_2, y_2, z_2)$ are $\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2}$.

3. Find the coördinates of the points which trisect the segment $(1, -2, 4), (-3, 4, 5)$.

4. Show that the medians of the triangle whose vertices are the points $(1, 1, 0), (2, -1, 1)$, and $(3, 2, -1)$ meet in the point $(2, \frac{2}{3}, 0)$.

5. Show that the medians of any triangle meet in a point.

SUGGESTION. Let the coördinates of the vertices be (x_1, y_1, z_1), (x_2, y_2, z_2), and (x_3, y_3, z_3). The medians meet in the point

$$\frac{x_1 + x_2 + x_3}{3}, \quad \frac{y_1 + y_2 + y_3}{3}, \quad \frac{z_1 + z_2 + z_3}{3}.$$

This point is the center of gravity of the triangle.

6. Show that the lines joining the middle points of opposite edges of a tetrahedron pass through the same point and are bisected by that point.

7. Show that the lines joining the vertices of any tetrahedron to the point of intersection of the medians of the opposite face meet in a point which is three fourths of the distance from each vertex to the opposite face. This point is called the center of gravity of the tetrahedron.

8. Find the ratio in which the point $(2, -1, 5)$ divides the segment $(4, 13, 3)$, $(3, 6, 4)$; the point $(2, -2, -6)$ divides the segment $(4, -5, -12)$, $(-2, 4, 6)$; the point $(2, 1, 4)$ divides the segment $(-3, 4, 2)$, $(7, -2, 6)$.

CHAPTER XII

LOCI AND THEIR EQUATIONS

153. Surfaces and curves. In space there are two kinds of loci to be considered. If a point moves according to a given law, it will, in general, describe a **surface**. Thus, if a point moves so as to be always at a given distance from a fixed point, it will describe a sphere whose center is the fixed point and whose radius is the given distance.

If a point moves so as to satisfy simultaneously two independent laws, it will, in general, describe a **line**, straight or curved. Thus, if a point moves so as to be at a fixed distance from the point A and at the same time at a fixed distance from the point B, it will describe the circle of intersection of the two spheres whose centers are at A and B and whose radii are the given fixed distances.

154. Equations of loci. When the law governing the motion of a point is expressed in terms of the coördinates of the point, the resulting equation is called the **equation of the surface** described by the point. The surface is called the **locus of the equation.**

Similarly, when a moving point is governed by two independent laws and these laws are expressed in terms of the coördinates of the moving point, the resulting equations are called the **equations of the curve** described by the point. The curve is called the **locus of the equations.**

As in plane geometry, two fundamental problems arise: First, given the law (or laws) governing the motion of a point, to find the equation (or equations) of the locus; and second, given the equation (or equations), to find the properties of the locus. These problems will be illustrated in the succeeding pages.

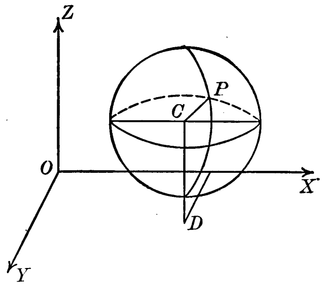

FIG. 125

155. The sphere. Let $C \equiv (a, b, c)$ be the center of a sphere whose radius is r, and $P \equiv (x, y, z)$, any point on the sphere. The length of CP is then,

$$r = \sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2}.$$

Hence, the equation of the sphere is

$$(x-a)^2 + (y-b)^2 + (z-c)^2 = r^2. \tag{1}$$

When the binominal squares are expanded, the equation has the form

$$x^2 + y^2 + z^2 + Ax + By + Cz + D = 0, \tag{2}$$

where A, B, C, and D are constants depending upon the coördinates of the center and the radius.

Conversely, an equation of the form (2) represents a sphere. For it can be written in the form

$$\left(x + \frac{A}{2}\right)^2 + \left(y + \frac{B}{2}\right)^2 + \left(z + \frac{C}{2}\right)^2 = \frac{A^2}{4} + \frac{B^2}{4} + \frac{C^2}{4} - D,$$

and hence represents a sphere whose center is $\left(-\frac{A}{2}, -\frac{B}{2}, -\frac{C}{2}\right)$ and whose radius is $\sqrt{\frac{A^2}{4} + \frac{B^2}{4} + \frac{C^2}{4} - D}$.

The sphere is real, so long as the expression under the radical is positive; it will be a null-sphere, or a point, when the expression under the radical is zero; and it will be an imaginary sphere when the expression under the radical is negative.

EXERCISES

1. Write the equation of a sphere whose center is $(5, -2, 3)$ and whose radius is 1; also of a sphere whose center is $(2, -3, -6)$ and which passes through the origin. What is the equation of a sphere whose center is on the Z-axis, has the radius a, and passes through the origin?

2. Which of the following spheres are real, which are null-spheres, and which are imaginary spheres? Find the center and radius of the real spheres.

(*a*) $x^2 + y^2 + z^2 - 2x + 6y - 8z + 22 = 0.$

(*b*) $x^2 + y^2 + z^2 + 10x - 4y + 2z + 5 = 0.$

(*c*) $x^2 + y^2 + z^2 + 4x + 4y + 6z + 1 = 0.$

(*d*) $x^2 + y^2 + z^2 + 6x = 0.$

(*e*) $x^2 + y^2 + z^2 + 4x + y + 5z + 21 = 0.$

3. Find the equation of the sphere passing through the four points (0, 0, 0), (2, 8, 0), (5, 0, 15), (– 3, 8, 1).

SUGGESTION. Substitute the coördinates of the given points in equation (2) and solve the resulting equations for the unknown coefficients A, B, C, D.

4. Find the equation of the sphere passing through the four points (2, 5, 14), (2, 10, 11), (– 2, 5, – 14), (2, – 10, – 11).

5. Find the equation of each of the two spheres whose center is at the origin and which touch the sphere

$$x^2 + y^2 + z^2 - 8x - 6y + 24z + 48 = 0.$$

156. Surfaces of revolution. When a curve in the XZ-plane is rotated about the X-axis, it describes a surface of revolution. Every point on the curve, as Q, describes a circle whose plane is perpendicular to the X-axis and whose radius is the ordinate DQ.

Let the coördinates of Q be $OD = x$ and $DQ = z'$, and the equation of the curve MQR be $f(x, z') = 0$. Now

$$z' = DQ = DP = \sqrt{z^2 + y^2}.$$

Hence, we have the following conclusion:

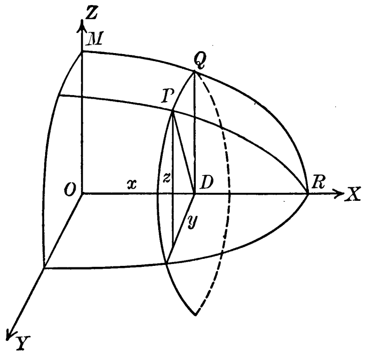

FIG. 126

To find the equation of the surface described by MQR, replace z' in the equation $f(x, z') = 0$ by its value $\sqrt{z^2 + y^2}$.

By a similar consideration we may find the equation of a surface of revolution obtained by rotating a given curve about either of the other axes.

EXERCISES

1. Find the equation of the ellipsoid obtained by rotating the ellipse $\frac{x^2}{a^2} + \frac{z'^2}{b^2} = 1$ about the X-axis. The ellipsoid of revolution is called the **prolate spheroid** when $a > b$, and the **oblate spheroid** when $a < b$. Explain, by familiar examples, the difference in form.

2. Find the equation of the paraboloid of revolution by rotating the parabola $z'^2 = 4\,px$ about the X-axis.

3. If the hyperbola $\frac{x^2}{a^2} - \frac{z'^2}{b^2} = 1$ and its conjugate $\frac{x^2}{a^2} - \frac{z'^2}{b^2} = -1$ are rotated about the X-axis, how will the two surfaces obtained differ? Find their equations. The first is called an **hyperboloid of two sheets** and the second, an **hyperboloid of one sheet.**

4. Show that if a curve in the XY-plane, whose equation is $f(x, y) = 0$, is rotated about the X-axis, the equation of the resulting surface is found by replacing y by $\sqrt{y^2 + z^2}$; and if the curve is rotated about the Y-axis, the equation of the resulting surface is obtained by replacing x by $\sqrt{x^2 + z^2}$.

5. What is the equation of the surface obtained by rotating the parabola $y^2 = 4\,px$ about the X-axis? about the Y-axis? How do the two surfaces differ?

157. Cylinders. If a straight line moves so as to be always parallel to one of the coördinate axes and, at the same time, intersects a curve lying in the plane of the other two axes, it describes a cylinder whose equation is the same as the equation of the curve. For, suppose the moving line is always parallel to the Z-axis and meets a curve in the XY-plane; then the x- and y-coördinates of any point on this line will be the same as the x- and y-coördinates of the point where the line meets the curve, and will consequently satisfy the equation of the curve whatever be the value of z. Moreover, the x- and y-coordinates of a point not on the cylinder cannot satisfy the equation of the curve. Therefore the equation of the curve, regarded as the equation of a locus in space, represents a cylinder parallel to the Z-axis. Similarly we may obtain the equation

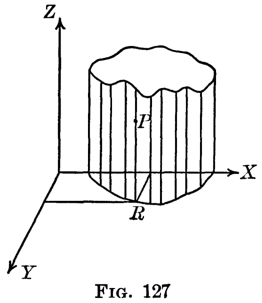

FIG. 127

of a cylinder parallel to any other axis. Hence, we have the conclusion:

Any equation in two of the three variables x, y, z represents a cylinder parallel to one of the coördinate axes.

EXERCISES

1. The following equations represent loci in space. Interpret them and draw the figures. (*a*) $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$; (*b*) $z^2=4px$; (*c*) $z+3y=6$.

2. A point moves so as to satisfy simultaneously the two equations $\frac{x}{2}+\frac{y}{3}=1$ and $\frac{y}{4}+\frac{z}{5}=1$. Plot its locus in space.

3. A point moves so as to satisfy simultaneously the two equations $x^2+y^2=4$ and $\frac{x}{3}+\frac{z}{2}=1$. Plot its locus in space.

4. Show that a point can move so as to satisfy simultaneously the three equations $3x+2y=6$, $5y+4z=20$, and $8z-15x=10$.

158. The right circular cone. When the straight line $z=mx$ is revolved about the X-axis, it generates a right circular cone whose vertex is at the origin and whose axis is the X-axis. Every generator of the cone makes an angle with the axis whose tangent is m. By Art. 156, the equation of this cone is

$$y^2+z^2=m^2x^2.$$

159. Plane sections of a right circular cone. Let APB (Fig. 128) be the curve common to the cone and any plane, as $AFPB$. Inscribe a sphere in the cone touching the cutting plane at F, and the cone along the small circle LES. The cutting plane and the plane of the circle meet in the line DD_1. Through P, any point of the curve APB, draw the generator of the cone VP, meeting the small circle in E. From P drop the perpendicular PK upon the plane LES, and draw PR perpendicular to DD_1. The angle $PRK=\alpha$ is the angle between the cutting plane and the plane LES and is therefore constant for all positions of P. The angle $PEK=\beta$ is also constant for all positions of P. The lines PF and PE are equal in length, since they are tangents to the sphere from an external point. Hence,

$$\frac{PF}{PR}=\frac{PE}{PR}=\frac{PK}{PR}\div\frac{PK}{PE}=\frac{\sin\alpha}{\sin\beta};$$

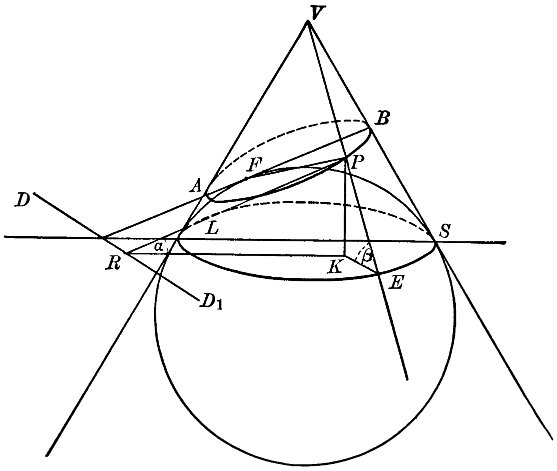

FIG. 128

and the curve APB is therefore a conic having one focus at F, the corresponding directrix being DD_1 (Art. 94, property A).

The conic will be an ellipse when $\alpha < \beta$, a parabola when $\alpha = \beta$; *i.e.* when the cutting plane is parallel to one of the generators of the cone, and an hyperbola when $\alpha > \beta$. In the figure, α is less than β and the section of the cone is therefore an ellipse.

EXERCISES

1. The equation of a right circular cone in spherical coördinates is $\theta = \text{const.}$ By means of the relations, Art. 143, exercise 1, transform this equation to rectangular coördinates.

2. Rotate the straight line $\frac{x}{2} + \frac{z}{3} = 1$ about the Z-axis and thus obtain the equation of a right circular cone whose vertex is at the point (0, 0, 3).

3. From Fig. 128, show how to locate the second focus of the section of the cone and its corresponding directrix.

4. The cone in example 2 is cut by a plane parallel to the Y-axis and meeting the XZ-plane in the line $\frac{x}{3} + \frac{z}{2} = 1$. Find the coördinates of the foci of the ellipse which this plane cuts from the cone.

5. The equation $\frac{x^2}{a^2} + \frac{y^2}{a^2} - \frac{z^2}{b^2} = 0$ represents a right circular cone. Write the equation of the straight line which describes this cone and tell about which axis it is revolved.

CHAPTER XIII

THE PLANE AND THE STRAIGHT LINE IN SPACE

160. The normal form of the equation of a plane. Let p denote the length of the perpendicular from the origin to the plane, and α, β, γ, the direction angles of this perpendicular. If L is the foot of the perpendicular, then any point, as P, will be in the plane if the angle OLP is a right angle; *i.e.* if the projection of the segment OP upon OL is equal to p. But the projection of OP upon OL is equal to the projection of the broken line $ODEP$ upon OL (Art. 149). Let the coördinates of P be $OD = x$, $DE = y$, and $EP = z$; then

$$x \cos \alpha + y \cos \beta + z \cos \gamma = p \qquad (1)$$

is the equation sought. It is called the **normal form** of the equation because it is expressed in terms of the perpendicular from the origin.

It follows that the equation of a plane is of first degree in the variables.

We shall now show that, conversely, every equation of the first degree in the variables x, y, z, is the equation of a plane.

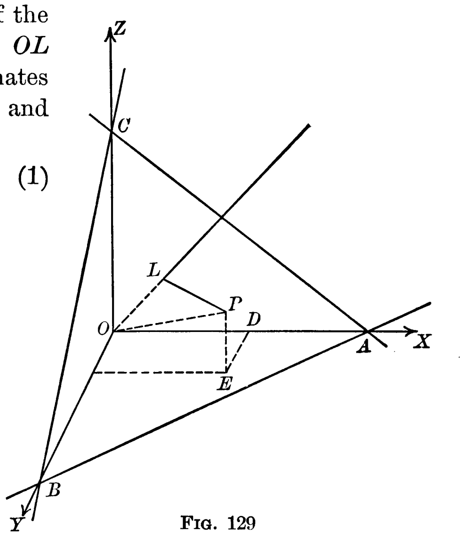

FIG. 129

For, let

$$Ax + By + Cz + D = 0 \qquad (2)$$

be any equation of the first degree in x, y, and z. Now if the coördinates of a point P satisfy this equation, they will still satisfy it after each of the coefficients A, B, C, D is multiplied by any constant k. The constant k can be chosen so that the equation $kAx + kBy + kCz = -kD$ will coincide with equation (1), term for term; that is, so that

$$kA = \cos\alpha,$$
$$kB = \cos\beta,$$
$$kC = \cos\gamma,$$
$$kD = -p.$$

Squaring and adding the first three of these equations, we have

$$k^2(A^2 + B^2 + C^2) = 1; \text{ (Art. 147)}$$

and therefore $k = \dfrac{1}{\pm\sqrt{A^2 + B^2 + C^2}}$. In order that p may be positive, the sign of the radical must be opposite to the sign of D. With this value of k, equation (2) agrees in form with equation (1) But (1) is the equation of a plane; therefore (2) is the equation of a plane for which

$$\cos\alpha = \frac{A}{\pm\sqrt{A^2 + B^2 + C^2}},$$

$$\cos\beta = \frac{B}{\pm\sqrt{A^2 + B^2 + C^2}},$$

$$\cos\gamma = \frac{C}{\pm\sqrt{A^2 + B^2 + C^2}},$$

$$p = \frac{-D}{\pm\sqrt{A^2 + B^2 + C^2}}.$$

The distances from the origin to the points where a plane meets the coördinate axes are called **the intercepts.** The lines in which a plane meets the coördinate planes are called **the traces.**

EXERCISES

1. Construct the planes and find their equations, for which (*a*) $\alpha = \frac{\pi}{4}$, $\beta = \frac{\pi}{3}$, $\gamma = \frac{\pi}{3}$, $p = 4$; (*b*) $\alpha = \frac{2\pi}{3}$, $\beta = \frac{3\pi}{4}$, $\gamma = \frac{\pi}{3}$, $p = 6$; (*c*) $\cos\alpha : \cos\beta : \cos\gamma = 6 : -2 : 3$, $p = 8$; (*d*) $\cos\alpha : \cos\beta : \cos\gamma = -2 : -1 : -2$, $p = 5$.

2. Find the equation of the plane such that the foot of the perpendicular from the origin to the plane is the point (*a*) $(3, -2, 6)$; (*b*) $(2, -5, 1)$; (*c*) $(3, 4, -2)$.

3. Reduce the following equations to normal form and find α, β, γ, and p.

(*a*) $6x - 3y + 2z - 7 = 0.$ (*b*) $x - \sqrt{2}y + z + 8 = 0.$
(*c*) $x - 4y - 2z - 3 = 0.$ (*d*) $x - 2y - 3 = 0.$

4. Find the intercepts and equations of the traces of the following planes.

(*a*) $2x + 5y - 3z - 4 = 0.$ (*b*) $x - y - z + 10 = 0.$ (*c*) $3x - y + z = 0.$

5. Find the area of the triangle which the coördinate planes cut from the plane $2x + 2y + z - 12 = 0$.

161. Intercept form of equation. Let the x-, y-, and z-intercepts of a plane be a, b, and c respectively; then (Fig. 126) the plane passes through the three points $A \equiv (a, 0, 0)$, $B \equiv (0, b, 0)$, and $C \equiv (0, 0, c)$. Since the equation of the plane is of the form

$$Ax + By + Cz + D = 0,$$

this equation must be satisfied by the coördinates of the points A, B, and C. Hence,

$$Aa + D = 0,$$
$$Bb + D = 0,$$
$$Cc + D = 0,$$

from which $A = -\frac{D}{a}$, $B = -\frac{D}{b}$, and $C = -\frac{D}{c}$. Substituting and reducing, the required equation is

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1.$$

162. Equation of a plane through three given points. If a plane is required to pass through three fixed points, the coördinates of these points must satisfy the general equation

$$Ax + By + Cz + D = 0,$$

and there are thus three equations from which to determine three of the unknown coefficients A, B, C, D in terms of the fourth. Substituting the three coefficients thus determined in the general equation gives the equation of the plane through the three given points.

EXERCISES

1. Write the equation of each of the planes having the following intercepts and find the length of the perpendicular from the origin upon each:

(*a*) 3, 1, 2. (*b*) -1, -2, 3. (*c*) 4, -2, 5. (*d*) -5, 2, -3.

2. Find the equation of the plane passing through the points (1, 0, 2), (0, 3, 4), and (− 1, 5, 0). Find the intercepts and the perpendicular from the origin.

3. Why will not the three points (1, 1, 2), (3, − 1, 3), and (5, − 3, 4) determine a plane? What are the direction cosines of the segments which join the first point to each of the other two?

4. From each of the points (2, 3, 0), (− 2, − 3, 4), and (0, 6, 0) drop perpendiculars to the *XZ*-plane. What are the coördinates of the feet of these perpendiculars? What is the area of the triangle formed in the *XZ*-plane? Drop perpendiculars to each of the other coördinate planes and compute the areas of the triangles formed in each. These triangles are called the **projections** of the space triangle upon the coördinate planes.

163. Determinant form of the equation. If a plane is required to pass through three given points (x_1, y_1, z_1), (x_2, y_2, z_2), and (x_3, y_3, z_3), the general equation

$$Ax + By + Cz + D = 0 \tag{1}$$

must be satisfied by the coördinates of these points. Hence the following equations hold,

$$Ax_1 + By_1 + Cz_1 + D = 0, \tag{2}$$

$$Ax_2 + By_2 + Cz_2 + D = 0, \tag{3}$$

$$Ax_3 + By_3 + Cz_3 + D = 0. \tag{4}$$

But in order that the four equations (1) to (4) may be satisfied by other than zero values of A, B, C, and D, it is necessary and sufficient that the determinant of their coefficients shall vanish; that is, we must have

$$\begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} = 0.$$

This equation is of the first degree in the variables x, y, z; and it is clearly satisfied by the coördinates of the given points. Therefore it is the equation of the plane passing through these points.

EXERCISES

1. Using the determinant form, find the equation of the plane which passes through the points (2, 3, 0), (– 2, – 3, 4), and (0, 6, 0).

2. In the same way, find the equation of the plane passing through the points (1, 1, – 1), (– 2, – 2, 2), and (1, – 1, 2).

3. Show that the direction cosines of the normal to a plane passing through three given points are proportional to the cofactors corresponding to x, y, and z in the determinant form of its equation.

4. Show that the cofactors corresponding to x, y, and z are proportional to the areas of the projections of the triangle whose vertices are (x_1, y_1, z_1), (x_2, y_2, z_2), and (x_3, y_3, z_3), upon the coördinate planes.

164. Perpendicular distance from a plane to a point. Given the equation of the plane ABC and the coördinates of the point $P_1 \equiv (x_1, y_1, z_1)$, it is required to find the length of the perpendicular DP_1, where D is a point in the plane ABC and P_1 lies outside of this plane. If the equation of the plane is not in the normal

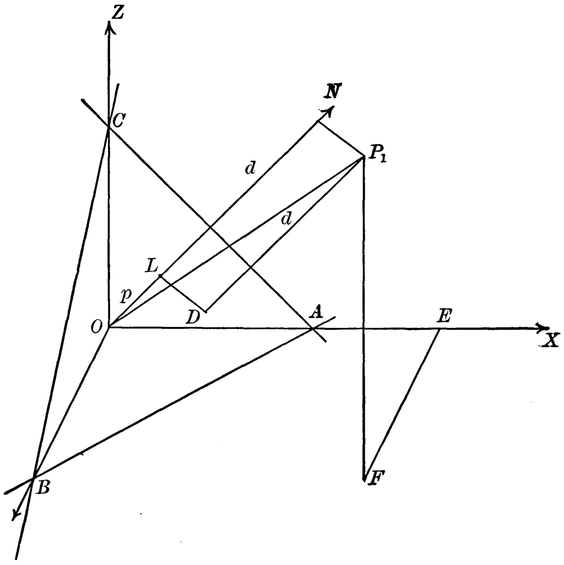

FIG. 130

form, reduce it to that form (Art. 160) so that the equation is

$$x \cos \alpha + y \cos \beta + z \cos \gamma - p = 0, \tag{1}$$

where the direction angles and the perpendicular are known.

Let d be the length of the required perpendicular, so that the projection of OP_1 upon ON is equal to $p + d$. But this projection is equal to the projection of the broken line $OEFP_1$ upon ON. Hence,

$$p + d = x_1 \cos \alpha + y_1 \cos \beta + z_1 \cos \gamma,$$

or

$$\boldsymbol{d = x_1 \cos \alpha + y_1 \cos \beta + z_1 \cos \gamma - p.}$$

The length of the perpendicular is therefore equal to the result of substituting the coördinates of the given point in the left member of (1). The result of the substitution will be negative if the point lies on the same side of the plane as the origin; and positive if the point and the origin are on opposite sides of the plane.

EXERCISES

1. Find the distance from the plane $6x - 3y + 2z - 10 = 0$ to the point (4, 2, 10). From the plane $4x + 3y + 12z + 6 = 0$ to the point (9, -1, 0). State if the point and the origin are on the same side, or on opposite sides, of the plane.

2. Find the length of the altitude of the tetrahedron from the vertex (2, 0, 1) to the plane of the vertices (0, 5, -4), (0, 3, 1), and (2, -7, 1).

3. The x- and y-intercepts of a plane are 3 and 4, respectively, and the plane touches a sphere whose center is at the origin and whose radius is 2. Find the equation of the plane.

4. Find the volume of the tetrahedron whose vertex is the point (5, 5, 6) and whose base is the triangle cut from the plane $x + 2y + 5z - 10 = 0$ by the coördinate planes.

5. Find the volume of the tetrahedron whose vertices are (3, 4, 0), (4, -1, 0), (1, 2, 0), and (6, -1, 4). Of the tetrahedron whose vertices are (3, 0, 0), (0, 1, 0), (0, 0, 5), (5, -2, 4).

6. Find the locus of points which are equally distant from the two planes $x - 2y + 3z - 4 = 0$ and $2x + 3y - z - 5 = 0$.

7. What is the equation of the locus of a point which is equally distant from the origin and from the plane $x + y + z - 1 = 0$?

165. Angle between two planes. The angle between two planes is equivalent to the angle between the perpendiculars to these planes. Let the equations of two planes be

$$A_1x + B_1y + C_1z + D_1 = 0 \text{ and } A_2x + B_2y + C_2z + D_2 = 0.$$

The direction cosines of the perpendiculars to these planes are given in Art. 160. Hence, Art. 150, we have

$$\cos\theta = \frac{A_1A_2 + B_1B_2 + C_1C_2}{\pm\sqrt{A_1^2 + B_1^2 + C_1^2} \cdot \pm\sqrt{A_2^2 + B_2^2 + C_2^2}},$$

the signs of the radicals being chosen as in Art. 160.

It follows from this formula that: two planes will be perpendicular if, and only if,

$$A_1A_2 + B_1B_2 + C_1C_2 = 0.$$

For only then can $\cos\theta$ be equal to zero, and consequently $\theta = 90°$.

Two planes will be parallel if, and only if,

$$\frac{A_1}{A_2} = \frac{B_1}{B_2} = \frac{C_1}{C_2}.$$

For only then will the perpendiculars to the planes be parallel to each other.

EXERCISES

1. The three planes $x + y + z - 2 = 0$, $x - y - 2z = 4$, and $2x + y - z = 2$ meet in a point forming a trihedral angle. Find the vertex of the angle and the three dihedral angles.

2. Find the equation of the plane which passes through the points (0, 3, 0) and (4, 0, 0) and is perpendicular to the plane $4x - 6x - z - 12 = 0$.

3. Find the equation of the plane which passes through the point (1, 2, 4) and is perpendicular to each of the planes $2x - 3y - z + 2 = 0$ and $x - y + 2z - 4 = 0$.

4. Find the equation of the plane that is perpendicular to the segment joining (3, 4, − 1) to (− 3, 6, 1) at its middle point.

5. Find the equation of the plane which passes through the point (3, − 3, 0) and is parallel to the plane $3x - y + z - 6 = 0$.

166. Pencil of planes with a common axis. The system of planes passing through the line of intersection of two given planes

$$A_1x + B_1y + C_1z + D_1 = 0 \text{ and } A_2x + B_2y + C_2z + D_2 = 0$$

is called a pencil of planes with a common axis, or a **coaxial pencil.** The pencil is represented by the equation

$$A_1x + B_1y + C_1z + D_1 + k(A_2x + B_2y + C_2z + D_2) = 0, \quad (1)$$

where k is an arbitrary constant. For, it is clear that every point whose coördinates satisfy both the given equations will be a point lying on the locus of (1); and, since (1) is of first degree in the variables, it is the equation of a plane. Therefore (1) is the equation of a plane passing through the line of intersection of the given planes, whatever value is given to k.

167. Pencil of planes with a common vertex. The system of planes passing through the point $P_1 \equiv (x_1, y_1, z_1)$ is called a **pencil of planes with a common vertex**, the point P_1 being the vertex. It is represented by the equation

$$A(x - x_1) + B(y - y_1) + C(z - z_1) = 0,$$

where A, B, and C are arbitrary constants. For this equation is the equation of a plane, whatever be the values of A, B, and C, and it is clearly satisfied by the coördinates of P_1. Therefore it represents a plane passing through P_1.

EXERCISES

1. A plane passes through the point $(3, 2, -1)$ and is parallel to the plane $7x - y + z - 14 = 0$. Find its equation.

2. Determine k so that the plane $x + ky - 2z - 9 = 0$ shall pass through the point $(1, 4, -3)$. So that it shall be parallel to the plane $3x - 4y + 2z - 5 = 0$. So that it shall be perpendicular to the plane $5x - 3y - z - 2 = 0$.

3. Find the equation of the plane which passes through the intersection of the planes $2x - 3y - z - 6 = 0$ and $x + y + z = 5$, and (a) passes through the point $(3, -2, 1)$; (b) is perpendicular to the plane $x - y - z + 2 = 0$.

4. Find the equations of the planes which pass through the intersection of the planes $x - y - 3z - 4 = 0$ and $x + y + 5z - 6 = 0$, and are perpendicular respectively to each of the coördinate planes.

5. Find the equations of the planes which are parallel to the plane $6x - 5y - 3z - 2 = 0$ and which touch a sphere of radius 3 whose center is at the origin.

6. Find the equation of the plane which is parallel to the plane $5x - 3y - 7z - 8 = 0$ and such that the point $(5, -1, 2)$ lies midway between the two planes.

7. Find the equation of a plane through the point $(2, -3, 0)$ and having the same trace upon the XZ-plane as the plane $x - 3y + 7z - 2 = 0$.

8. Find the equation of the plane parallel to the plane

$$2x + y + 2z + 5 = 0,$$

and forming a tetrahedron of unit volume with the coördinate planes.

9. Find the equation of the plane parallel to the plane

$$5x + 3y + z - 7 = 0$$

such that the sum of its intercepts is 23.

10. Find the equation of the plane having the trace $x + 3y - 2 = 0$, and forming a tetrahedron of volume $\frac{4}{3}$ with the coördinate planes.

168. The equations of a straight line in space. If $P_1 \equiv (x_1, y_1, z_1)$ and $P_2 \equiv (x_2, y_2, z_2)$ are any two points in space, then the coördinates of the point dividing the segment P_1P_2 in the ratio $\frac{P_1P}{PP_2} = r$, are (Art. 152),

$$x = \frac{x_1 + rx_2}{r+1},\quad y = \frac{y_1 + ry_2}{r+1},\quad z = \frac{z_1 + rz_2}{r+1}. \tag{1}$$

When r is allowed to vary, these equations give the coördinates of a variable point on the line P_1P_2 and are, therefore, the **parametric equations** of the line P_1P_2, r being the parameter.

From equations (1), or from the figure, Art. 152, it follows easily that

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}. \tag{2}$$

These equations are called the **two-point form** of the equations of the line P_1P_2.

Since the direction cosines of P_1P_2 are proportional to the projections upon the coördinate axes (Art. 146), we have, from equations (2),

$$\frac{x - x_1}{\cos\alpha} = \frac{y - y_1}{\cos\beta} = \frac{z - z_1}{\cos\gamma}. \tag{3}$$

These equations are called the **symmetric form** of the equations of the line P_1P_2.

As an example, the equations of the straight line through the points (3, -2, 1) and (4, 5, -6) are, in the parametric form,

$$x=\frac{3+4r}{r+1},\ y=\frac{-2+5r}{r+1},\ z=\frac{1-6r}{r+1}.$$

In the two-point form, they are

$$\frac{x-3}{1}=\frac{y+2}{7}=\frac{z-1}{-7}.$$

The direction cosines are $\cos\alpha=\frac{1}{\sqrt{99}}$, $\cos\beta=\frac{7}{\sqrt{99}}$, $\cos\gamma=\frac{-7}{\sqrt{99}}$.

EXERCISES

1. Find the equations of the lines joining the following pairs of points:

(*a*) (0, 0, -2) to (3, -1, 0). (*b*) (-1, 3, 2) to (2, -2, 4).
(*c*) (2, -3, 1) to (2, -3, -1).

2. In the preceding exercise, find the coördinates of the points where each line meets the coördinate planes.

3. Find the direction cosines of each of the lines in exercise 1.

4. Find the equations of the line through the point (-1, 2, -3) if

(*a*) $\alpha=60°$, $\beta=60°$, $\gamma=45°$.
(*b*) $\alpha=120°$, $\beta=60°$, $\gamma=135°$.
(*c*) $\cos\alpha=\frac{\sqrt{3}}{2}$, $\cos\beta=\frac{1}{2}$ $\cos\gamma=0$.

Show that the given values are possible in each case and plot the line.

5. Find the equations of the line through the origin and equally inclined to the axes.

169. The projecting planes of a line. The planes drawn through a given line and perpendicular to each of the coördinate planes in turn, are called the **projecting planes** of the line.

The equation of a projecting plane can contain only two of the three variables x, y, z (Art. 157). Hence the equations of the projecting planes can be found from equation (2) or (3) of the preceding article, by neglecting one of the ratios involved. Thus, for example, the equation of the projecting plane perpendicular to the XY-plane is

$$\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}.$$

For this equation represents a plane parallel to the Z-axis, and it is satisfied by the coördinates of the points P_1 and P_2. Similarly, the equations of the other projecting planes can be found.

The equation of any plane through the given line is,

$$\frac{x-x_1}{x_2-x_1}-\frac{y-y_1}{y_2-y_1}=k\left(\frac{x-x_1}{x_2-x_1}-\frac{z-z_1}{z_2-z_1}\right).$$

For, this equation is linear in x, y, z and is therefore the equation of a plane; moreover it is satisfied by the coördinates of P_1 and P_2 irrespective of the value of the parameter k.

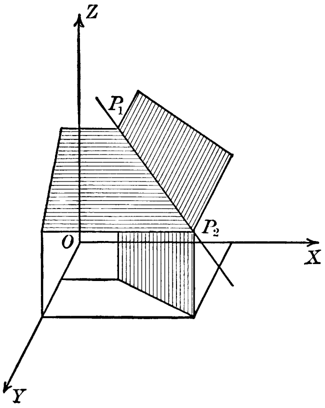

FIG. 131

170. The intersection of two planes. If a line is given as the intersection of two planes, its equations are the equations of the two planes considered as simultaneous equations. Thus, the equations

$$A_1x+B_1y+C_1z+D_1=0 \text{ and } A_2x+B_2y+C_2z+D_2=0 \qquad (1)$$

are the equations of the line of intersection of the two planes whose equations are those just written. The direction cosines of the perpendiculars to these planes are respectively proportional to A_1, B_1, C_1 and A_2, B_2, C_2; and, since the line of intersection is at right angles to both these perpendiculars, its direction cosines must satisfy the two equations

$$A_1\cos\alpha+B_1\cos\beta+C_1\cos\gamma=0,$$
$$A_2\cos\alpha+B_2\cos\beta+C_2\cos\gamma=0. \quad \text{(Art. 151)}$$

Hence, we have

$$\cos\alpha:\cos\beta:\cos\gamma=(B_1C_2-B_2C_1):(A_2C_1-A_1C_2):(A_1B_2-A_2B_1).$$

Therefore the equations of the line of intersection are

$$\frac{x-x_1}{B_1C_2-B_2C_1}=\frac{y-y_1}{A_2C_1-A_1C_2}=\frac{z-z_1}{A_1B_2-A_2B_1},$$

where (x_1, y_1, z_1) is any point on the line.

To find a point on the line, put one of the variables equal to zero in equations (1) and solve the resulting equations for the other two.

For example, consider the two planes

$$2x + 4y + 2z - 8 = 0,$$

and

$$5x + 6y + z - 16 = 0.$$

The direction cosines of their line of intersection must satisfy the two equations

$$2\cos\alpha + 4\cos\beta + 2\cos\gamma = 0,$$

and

$$5\cos\alpha + 6\cos\beta + \cos\gamma = 0.$$

Hence,

$$\cos\alpha : \cos\beta : \cos\gamma = -8 : 8 : -8 : = -1 : 1 : -1.$$

To find the coördinates of a point on the line, put $z = 0$ in the equations of the planes and solve the resulting equations for x and y. In this way we find that the point (2, 1, 0) lies on the line. Therefore the equations of the line are

$$\frac{x-2}{-1} = \frac{y-1}{1} = \frac{z}{-1}.$$

EXERCISES

1. What are the equations of the projecting planes of the line

$$\frac{x-2}{6} = \frac{y-3}{3} = \frac{z}{2}?$$

What is the equation of the plane passing through this line and through the origin? Through this line and through the point (1, − 2, 5)?

2. What are the equations of the line through the point (2, 5, 7) if $\cos\alpha = \frac{2}{3}$, $\cos\beta = \frac{\sqrt{5}}{3}$, and $\cos\gamma = 0$? If $\cos\alpha = \frac{1}{2}$, $\cos\beta = 0$, and $\cos\gamma = \frac{\sqrt{3}}{2}$? If $\cos\alpha = 0$, $\cos\beta = \frac{1}{3}$, and $\cos\gamma = 2\frac{\sqrt{2}}{3}$?

3. Find the equations of the projecting planes of the line

$$2x - 3y + z - 6 = 0,\quad x + y - 3z - 1 = 0$$

by eliminating x, y, and z in turn from these equations.

4. Find the direction cosines of the line

$$2x + 3y - 2z - 13 = 0,\quad 3x + 6y - 3z - 24 = 0.$$

5. Find the coördinates of the points in which the line

$$2x + 2y - 3z - 2 = 0,\quad 4x - y - z - 6 = 0$$

meets the coördinate planes.

6. Reduce the equations $x + 2y + 6z - 5 = 0$, $3x - 2y - 10z - 7 = 0$ to the symmetric form.

Eliminating in turn x and y from the given equations, we find the equations of two of the projecting planes to be

$$2y + 7z - 2 = 0 \text{ and } x - z - 3 = 0.$$

From the first, $z = \frac{-2(y-1)}{7}$; and from the second, $z = x - 3$. Hence we have,

$$\frac{x-3}{2} = \frac{y-1}{-7} = \frac{z}{2},$$

from which the symmetric form follows at once.

7. In the same way, reduce the equations

$$2x + 2y - 3z - 2 = 0,\ 4x - y - z - 6 = 0,$$

to the symmetric form.

8. Reduce the equations $x = mz + a$ and $y = nz + b$ to the symmetric form.

9. Find the direction cosines of the following lines:

(*a*) $4x - 5y + 3z = 3$, $4x - 5y + z + 9 = 0$.
(*b*) $2x + z + 5 = 0$, $x + 3z - 5 = 0$.
(*c*) $3x - y - 2z = 0$, $6x - 3y - 4z + 9 = 0$.

10. What are the equations of the line through the point $(2, 0, -2)$ and perpendicular to the lines

$$\frac{x-3}{2} = \frac{y}{1} = \frac{z+1}{2} \text{ and } \frac{x}{3} = \frac{y+1}{-1} = \frac{z+2}{2}\ ?$$

11. What are the equations of a line through the point $(2, 3, 4)$ if $\cos\alpha = \cos\beta = 0$?

171. Intersection of a line with a plane. The coördinates of the point of intersection of a line with a plane must satisfy the equations of the line and also the equation of the plane. Hence, to find these coördinates, solve the three equations simultaneously for x, y, and z.

When the equations of the line are not given, but the coördinates of two points on the line are known, a more expeditious method is to write the equations of the line in parametric form (Art. 168, (1)), substitute these values of x, y, and z in the equation of the plane, and solve for r, thus determining the ratio in which the required point divides the segment joining the given points. For example, to find the coördinates of the point in which the line joining the points $(1, -2, 0)$ and $(3, -4, 5)$ meets the plane $x - y + 4z + 2 = 0$, write the parametric equations of the line; viz.:

$$x = \frac{1 + 3r}{r + 1},$$

$$y = \frac{-2 - 4r}{r + 1},$$

and
$$z = \frac{5r}{r + 1};$$

and substitute these values of x, y, and z in the equation of the plane. In this way we find $r = -\frac{5}{29}$. Hence the point of intersection is $(\frac{14}{24}, -\frac{38}{24}, -\frac{25}{24})$.

The line whose direction angles are α, β, and γ will be parallel to the plane $Ax + By + Cz + D = 0$ if, and only if,

$$A \cos \alpha + B \cos \beta + C \cos \gamma = 0.$$

For only then will the line be at right angles to every perpendicular to the plane.

The line will be perpendicular to the plane if, and only if,

$$\frac{A}{\cos \alpha} = \frac{B}{\cos \beta} = \frac{C}{\cos \gamma}.$$

For only then will the line be parallel to every perpendicular to the plane.

EXERCISES

1. Find the coördinates of the point in which the line $x + 4y + 2z = 0$, $y - 3z - 7 = 0$ meets the plane $3x - 2y + z + 4 = 0$; the coördinates of the point in which the line $\frac{x-2}{3} = \frac{y-1}{2} = \frac{z-3}{-4}$ meets the plane $x + y + z - 2 = 0$; the coördinates of the point in which the line joining the points $(2, -3, 1)$, $(2, -2, 4)$ meets the plane $x - y - z - 5 = 0$.

2. Show that the line $\frac{x+3}{2} = \frac{y-4}{-7} = \frac{z}{3}$ is parallel to the plane

$$4x + 2y + 2z = 9.$$

3. Show that the line $\frac{x}{3} = \frac{y}{2} = \frac{z}{7}$ is perpendicular to the plane

$$3x + 2y + 7z = 8.$$

4. Show that the two straight lines $x - 2 = 2y - 6 = 3z$ and $4x - 11 = 4y - 13 = 3z$ meet in a point. Find the coördinates of this point and show that the two lines lie in the plane $2x - 6y + 3z + 14 = 0$.

5. Find the equations of the line passing through $(1, -6, 2)$ and perpendicular to the plane $2x - y + 6z = 0$.

6. Find the equations of the line passing through the point $(-2, 3, 2)$ which is parallel to each of the planes $3x - y + z = 0$ and $x - z = 0$.

7. Show that the six planes, each containing an edge of a tetrahedron and bisecting the opposite edge, meet in a point.

8. Prove that the six planes, each passing through the middle point of one edge of a tetrahedron and perpendicular to the opposite edge, meet in a point.

9. What is the equation of a plane passing through the point $(1, 3, -2)$ and perpendicular to the line

$$\frac{x-3}{2} = \frac{y-4}{5} = \frac{z}{-1}\,?$$

10. Find the equation of the plane determined by the parallel lines

$$\frac{x+1}{3} = \frac{y-2}{2} = \frac{z}{1} \quad \text{and} \quad \frac{x-3}{3} = \frac{y+4}{2} = \frac{z-1}{1}.$$

11. Find the equations of the line tangent to the sphere $x^2 + y^2 + z^2 = 9$ at the point $(2, -1, -2)$ and parallel to the plane $x + 3y - 5z - 1 = 0$.

12. What are the equations of the line passing through the point (x_1, y_1, z_1) and perpendicular to the plane $Ax + By + Cz + D = 0$?

13. What is the equation of the plane passing through the point (x_1, y_1, z_1) and perpendicular to the line

$$\frac{x-x_2}{a} = \frac{y-y_2}{b} = \frac{z-z_2}{c}\,?$$

CHAPTER XIV

EQUATIONS AND THEIR LOCI

172. Second fundamental problem. The two fundamental problems of solid analytic geometry were stated in Art. 154. The first of these has been illustrated in the preceding articles by finding the equations of certain well-known loci, such as the sphere, the right circular cone, the plane, the straight line. In this chapter we shall consider the second fundamental problem; that is, given an equation in the three variables x, y, z, to find the form and properties of the locus.

173. Construction of a surface from its equation. The following rules serve as a guide in sketching, or constructing, a surface from its equation.

(1) *Symmetry.* If the equation contains only even powers of one of the variables, the surface is symmetrical with respect to the coördinate plane from which that variable is measured. For example, if the equation contains only even powers of z, and the point (a, b, c) is on the surface, then the point $(a, b, -c)$ will also be on the surface. The XY-plane is then a **plane of symmetry.**

If the equation contains only even powers of two of the variables, the surface is symmetrical with respect to the coördinate axis along which the third variable is measured. For example, if the equation contains only even powers of y and z, the surface is symmetrical with respect to the XZ-plane and also with respect to the XY-plane, and hence with respect to their intersection, or the X-axis. The X-axis is then a **line of symmetry.**

If an equation contains only even powers of all three of the variables, the surface is symmetrical with respect to each of the coördinate planes and therefore with respect to their intersection, or the origin. The origin is then a **point of symmetry.**

(2) *Intercepts.* The length of the segments from the origin to the points where a surface meets the coördinate axes are called its **intercepts.** These are found by putting two of the variables equal to zero and solving the resulting equation for the third variable.

(3) *Traces.* The sections of a surface made by the coördinate planes are called the **traces of the surface.** The equations of the traces are found by putting each variable in turn equal to zero.

(4) *Plane sections parallel to the coördinate planes.* The equation of a surface and the equation $z = k$, a constant, are together the equations of the curve of intersection of the surface with a plane parallel to the XY-plane. A series of sections parallel to the XY-plane can be found by allowing k to vary. Similarly, sections parallel to the other coördinate planes can be found.

To construct a surface, it is customary to plot the traces upon the coördinate planes and a series of sections parallel to at least one of the coördinate planes.

174. The quadric surfaces, or conicoids. The locus of an equation of the second degree in x, y, z is called a **quadric surface,** or **conicoid.** It can be shown that any equation of the second degree in x, y, z is reducible by a proper transformation of the coördinate axes to one or the other of the two forms

$$\boldsymbol{Ax^2 + By^2 + Cz^2 = D}, \tag{1}$$

$$\boldsymbol{Ax^2 + By^2 = 2\,cz}. \tag{2}$$

If the coefficients in (1) are all different from zero, the surface is called a **central quadric,** the origin being the center. By the preceding article we see that the surface is symmetrical with respect to each of the coördinate planes, with respect to each of the coördinate axes, and with respect to the origin.

If the coefficients in (2) are all different from zero, the surface is called a **noncentral quadric.** The surface is clearly symmetrical with respect to the XZ- and YZ-planes and with respect to the Z-axis, but it is not symmetrical with respect to the XY-plane nor with respect to the X- and Y-axes.

If one or more of the coefficients in either (1) or (2) are zero, the surface is called a **degenerate quadric.**

175. The ellipsoid. If all the coefficients in (1) of the preceding article are positive, the equation can be written in the form

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1, \tag{1}$$

and the surface is called an **ellipsoid** (Fig. 132).

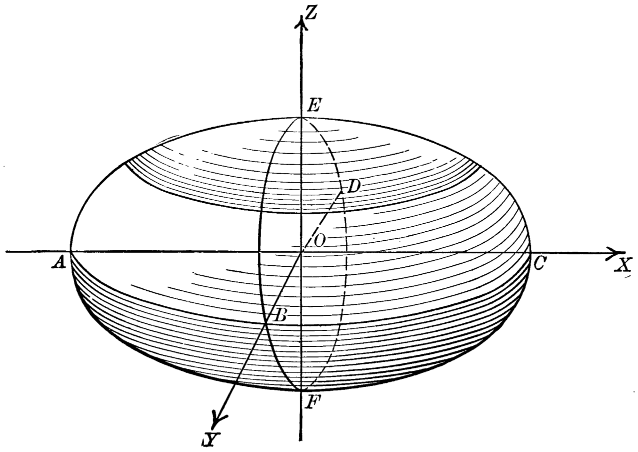

FIG. 132

The intercepts on the X-, Y-, Z-axes are respectively $\pm a$, $\pm b$, $\pm c$. The numbers a, b, c are the lengths of the semiaxes.

The traces on the coördinate planes are all ellipses, represented in the figure by $ABCD$, $BEDF$, and $AECF$. The equations of these traces are respectively

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,\quad \frac{y^2}{b^2}+\frac{z^2}{c^2}=1,\ \text{and}\ \frac{x^2}{a^2}+\frac{z^2}{c^2}=1.$$

Equation (1) can be written in the form

$$\frac{x^2}{a^2\left(1-\frac{z^2}{c^2}\right)}+\frac{y^2}{b^2\left(1-\frac{z^2}{c^2}\right)}=1,$$

from which we see that any section of the ellipsoid parallel to the XY-plane is an ellipse whose semiaxes are

$$a\sqrt{1-\frac{z^2}{c^2}},\ \text{and}\ b\sqrt{1-\frac{z^2}{c^2}}.$$

Hence, the section will be a real ellipse only when z is confined to the range

$$-c \leqq z \leqq c,$$

and reduces to a point if z is either $-c$ or $+c$.

Similarly, the sections parallel to the YZ-plane will be real only when x is confined to the range $-a \leqq x \leqq a$, and the sections parallel to the XZ-plane will be real only when y is confined to the range $-b \leqq y \leqq b$. The surface is therefore wholly inclosed within the parallelopiped whose edges are $2a$, $2b$, and $2c$.

If two of the numbers a, b, c are equal to each other, the surface is an ellipsoid of revolution (Art. 156; Ex. 1); and if all three are equal to the same number, it is a sphere (Art. 155).

EXERCISES

1. Construct the following ellipsoids:

(*a*) $4x^2+9y^2+16z^2=144$; (*b*) $x^2+16y^2+z^2=64$; (*c*) $16x^2+y^2+16z^2=64$.

2. Show that

$$\frac{(x-2)^2}{4}+\frac{(y-1)^2}{9}+\frac{(z-5)^2}{16}=1$$

is the equation of an ellipsoid whose center is (2, 1, 5).

3. In general,

$$\frac{(x-l)^2}{a^2}+\frac{(y-m)^2}{b^2}+\frac{(z-n)^2}{c^2}=1$$

is the equation of an ellipsoid whose center is the point (l, m, n).

4. Show that $x^2+2y^2+2z^2-2x+4y-8z+10=0$ is the equation of an ellipsoid whose center is the point $(1, -1, 2)$ and whose semiaxes are 1, $\frac{\sqrt{2}}{2}$, and $\frac{\sqrt{2}}{2}$. Is this surface an ellipsoid of revolution?

176. The hyperboloid of one sheet. If two of the coefficients, A, B, C in (1), Art. 174, are positive and one is negative, D being positive, the surface is called an **hyperboloid of one sheet.** Suppose C is the negative coefficient, the equation may then be written in the form

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1,$$

from which we see that the intercepts on the X- and Y-axes are respectively ± a and ± b; and that the surface does not meet the Z-axis.

The trace on the XY-plane is the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,$$

while the traces on the other two coördinate planes are the hyperbolas

$$\frac{x^2}{a^2}-\frac{z^2}{c^2}=1, \text{ and } \frac{y^2}{b^2}-\frac{z^2}{c^2}=1.$$

Any section parallel to the XY-plane is the ellipse

$$\frac{x^2}{a^2\left(1+\frac{z^2}{c^2}\right)}+\frac{y^2}{b^2\left(1+\frac{z^2}{c^2}\right)}=1,$$

which is clearly real for any value of z and increases indefinitely in size as z increases or diminishes indefinitely.

The sections parallel to the YZ-plane form a system of concentric hyperbolas (Art. 108), given by the equation

$$\frac{y^2}{b^2}-\frac{z^2}{c^2}=1-\frac{x^2}{a^2}.$$

If $x < a$, the transverse axis of the corresponding hyperbola is parallel to the Y-axis, and if $x > a$, the transverse axis of the hyperbola is parallel to the Z-axis. If $x = a$, the section of the surface is the two straight lines

$$\frac{y}{b}\pm\frac{z}{c}=0.$$

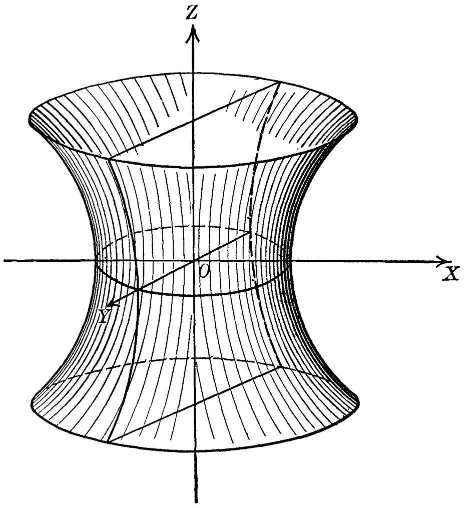

FIG. 133

Similarly, the sections parallel to the XZ-plane form a system of concentric hyperbolas. The form of the surface is shown in Fig. 133.

EXERCISES

1. Construct the following hyperboloids:

(*a*) $4x^2 + 9y^2 - 16z^2 = 144$; (*b*) $x^2 + y^2 - z^2 = 25$; (*c*) $x^2 + 16y^2 - z^2 = 64$.

2. Show that

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \text{ and } -\frac{x_2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

are the equations of hyperboloids of one sheet; the first surrounding the *Y*-axis, and the second, the *X*-axis.

3. Show that $$\frac{(x-1)^2}{4} + \frac{(y-3)^2}{9} - \frac{(z-2)^2}{1} = 1$$

is the equation of an hyperboloid of one sheet whose center is the point (1, 3, 2).

4. Show that $$\frac{(x-l)^2}{a^2} + \frac{(y-m)^2}{b^2} - \frac{(z-n)^2}{c^2} = 1$$

is the equation of an hyperboloid of one sheet whose center is the point (l, m, n).

5. Show that $3x^2 + 4y^2 - z^2 - 8y = 0$ is the equation of an hyperboloid of one sheet. Find the coördinates of its center.

6. Construct the surface whose equation is

$$x^2 - y^2 + 2z^2 - 6x + 2y + 4z + 9 = 0.$$

7. What are the equations of the planes parallel to the coördinate planes which cut the surface $9x^2 - y^2 + 9z^2 = 36$ in pairs of straight lines?

177. The hyperboloid of two sheets. If two of the coefficients A, B, C in (1), Art. 174, are negative and one is positive, D being positive, the surface is called an **hyperboloid of two sheets.** Suppose B and C are negative, the equation can then be written in the form

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1,$$

from which we see that the surface does not meet either the *Y*-axis or the *Z*-axis and consequently has no trace upon the *YZ*-plane. The traces upon the other coördinate planes are hyperbolas.

The sections of the surface parallel to the YZ-plane form a system of concentric ellipses given by the equation

$$\frac{y^2}{b^2}+\frac{z^2}{c^2}=\frac{x^2}{a^2}-1,$$

from which we see that the section will be a real ellipse only when x is confined to the range

$$-a \geqq x \geqq a.$$

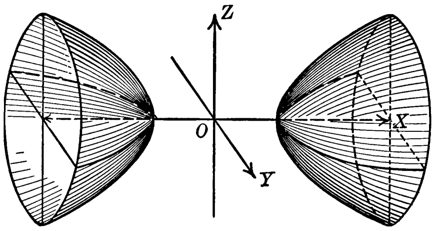

FIG. 134

The hyperboloid of two sheets is a surface of revolution if $b=c$. The form of the surface is shown in Fig. 134.

EXERCISES

1. Construct the hyperboloids $4\,x^2-9\,y^2-16\,z^2=1$ and $x^2-4\,y^2-4\,z^2=1$. Which is a surface of revolution ?

2. Show that

$$-\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1 \text{ and } -\frac{x^2}{a^2}-\frac{y^2}{y^2}+\frac{z^2}{c^2}=1$$

are equations of hyperboloids of two sheets ; the first surrounding the Y-axis, and the second, the Z-axis.

3. Show that $x^2-2\,y^2-4\,z^2-2\,x-8\,y-8=0$ and $y^2-x^2-2\,z^2+6\,x-2\,y-4\,z+6=0$ are equations of hyperboloids of two sheets. Find the coördinates of the center of each.

178. The elliptic paraboloid. If the coefficients A and B in (2), Art. 174, have the same sign, the surface is called an **elliptic paraboloid.** The equation can be written in the form

$$\boldsymbol{\frac{x^2}{a^2}+\frac{y^2}{b^2}=2\,cz,}$$

where c may be either positive or negative.

The trace on the XY-plane is a point, namely, the origin ; while the traces on the other coördinate planes are the parabolas

$$\frac{x^2}{a^2}=2\,cz \text{ and } \frac{y^2}{b^2}=2\,cz.$$

The sections of the surface parallel to the XY-plane form a system of concentric ellipses given by the equation

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=2\,cz,$$

from which we see that the section will be real only when c and z have the same sign. Hence the surface lies above or below the XY-plane according as c is positive or negative. The form of the surface is shown in Fig. 135, where c is supposed to be positive.

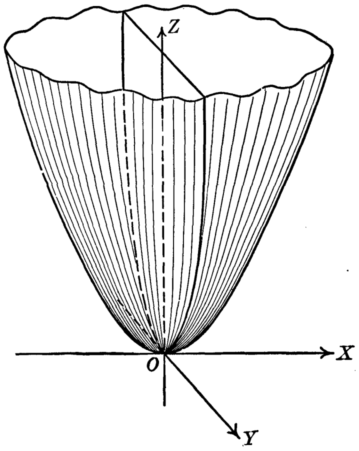

FIG. 135

179. The hyperbolic paraboloid. If the coefficients A and B in (2), Art. 174, are opposite in sign, the surface is called a **hyperbolic paraboloid**. Its equation can be written in the form

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=2\,cz.$$

The trace on the XY-plane is here a pair of straight lines

$$\frac{x}{a}\pm\frac{y}{b}=0$$

intersecting at the origin, and the sections parallel to the XY-plane form a system of concentric hyperbolas which recede from the trace on the XY-plane as z increases or diminishes. The transverse axis of one of these hyperbolas is parallel to the X-axis if c and z have the same sign, and parallel to the Y-axis if c and z have opposite signs.

The surface is saddle-shaped. A mountain pass between two solitary peaks resembles roughly a hyperbolic paraboloid (Fig. 136).

EXERCISES

1. Construct the following surfaces :

(a) $x^2+y^2=8\,z$. (b) $y^2+z^2=4\,x$.

(c) $x^2-4\,z^2=16\,y$. (d) $y^2-x^2=10\,z$.

2. Reduce each of the equations $x^2 + 2y^2 - 6x + 4y + 3z + 11 = 0$ and $z^2 - 3y^2 - 4x + 2z - 6y + 1 = 0$ to a standard form and determine the type of paraboloid of which each is the equation.

3. A point moves so that it is equidistant from two nonintersecting straight lines. Show that its locus is a hyperbolic paraboloid.

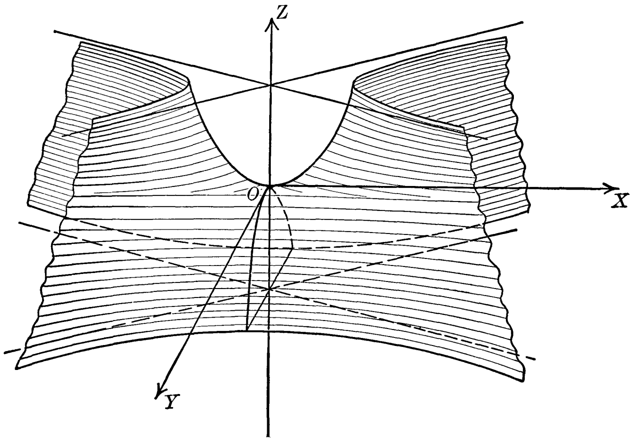

FIG. 136

4. Discuss the equations $z = xy$ and $z = x^2 + xy + y^2$. What are the loci of these equations ?

180. The quadric cone. If the constant term D in (1), Art. 174, is zero and the coefficients A, B, C are not all of the same sign, the locus of the equation is a **quadric cone**. Suppose that C is negative, and A and B are positive; the equation can then be written in the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0, \tag{1}$$

from which we see that the sections parallel to the XY-plane form a system of concentric ellipses which increase in size indefinitely from a point (when $z = 0$) as z increases or decreases indefinitely.

Again, if $P \equiv (x_1, y_1, z_1)$ is any point whose coördinates satisfy (1), we can prove that the line joining the origin to P lies entirely upon the surface. For the coördinates of any point on this line are clearly

$$x = rx_1,\ y = ry_1,\ z = rz_1,$$

where r is any number. But these coördinates satisfy (1) by virtue of the hypothesis that the coördinates of P satisfy (1). Therefore the cone may be generated by a line which rotates around the origin and intersects an ellipse whose axes are parallel to the X- and Y-axes, Fig. 137.

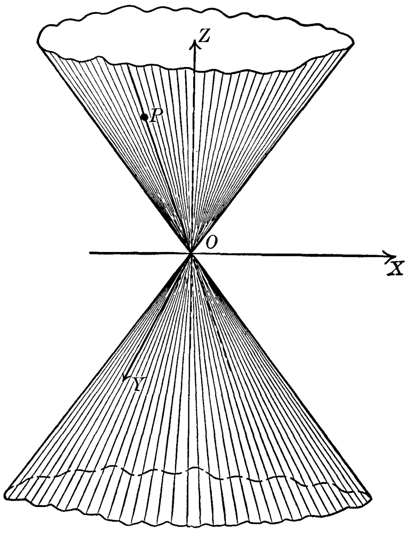

FIG. 137

181. Cylinders. If either (1) or (2), Art. 174, contains but two of the variables, the corresponding locus is a cylinder (Art. 157). The cylinders are therefore degenerate quadrics.

182. Pairs of planes. If an equation of the second degree is written with its right member equal to zero, and its left member is then the product of two expressions of the first degree in the variables, the corresponding locus is a pair of planes. For the equation is satisfied by the coördinates of any point which render either factor equal to zero. Thus,

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=0$$

is the equation of the pair of planes

$$\frac{x}{a}+\frac{y}{b}=0 \text{ and } \frac{x}{a}-\frac{y}{b}=0.$$

EXERCISES

1. Construct the cones whose equations are

$$(a)\ 9x^2-36y^2+4z^2=0, \text{ and } (b)\ 16x^2-4y^2-z^2=0.$$

2. If in (1), Art. 174, $D=0$ and A, B, and C are all of the same sign, what is the locus of the equation ?

3. Show that $x^2+4y^2-z^2-2x+8y+5=0$ is the equation of a cone whose vertex is the point $(1, -1, 0)$.

4. In general,

$$\frac{(x-l)^2}{a^2} \pm \frac{(y-m)^2}{b} \pm \frac{(z-n)^2}{c^2} = 0$$

is the equation of a cone whose vertex is the point (l, m, n).

5. Construct the cone whose equation is

$$x^2 + y^2 - 2z + 2y + 4z - 1 = 0.$$

6. Discuss the equations

(*a*) $4y^2 - 25 = 0$; (*b*) $2y^2 + 5z^2 = 0$; (*c*) $y^2 - x^2 - 4y + 6x - 5 = 0$.

What is the locus of each equation ?

7. Show that the left member of

$$x^2 - y^2 + z^2 + 2xz - 5x - y - 5z + 6 = 0$$

is divisible by $x + y + z - 2$. What is the locus of the given equation ?

183. Ruled surfaces. If a straight line moves according to a given law, it describes, or generates, a **ruled surface**. Thus, if a line moves so as to be constantly parallel to one of the coördinate axes, it describes a cylinder parallel to that axis. Again, if a line rotates about a fixed point and intersects a fixed curve, it generates a cone whose vertex is the fixed point. Cones and cylinders are ruled surfaces.

If the equations of a straight line contain a parameter k, then when k is allowed to vary, the line will move and thus describe a ruled surface. For example, let the equations of the line be

$$kx + y + z - k = 0 \text{ and } x - ky + kz + 1 = 0.$$

From the first of these equations we obtain

$$k = \frac{y+z}{1-x}, \tag{1}$$

and from the second,

$$k = \frac{1+x}{y-z}. \tag{2}$$

Therefore the coördinates of all the points that lie on the line must satisfy the equation

$$\frac{y+z}{1-x} = \frac{1+x}{y-z}, \tag{3}$$

or

$$x^2 + y^2 - z^2 = 1, \tag{4}$$

whatever the value of k. Conversely, any point whose coördinates satisfy (4) must lie on the given line for some value of k. For (4) is equivalent to (3) and the value of k is determined from either (1) or (2). The locus of (4) is an hyperboloid of one sheet. Therefore the given line generates this surface when k varies.

EXERCISES

1. Find the equation of the ruled surface generated by the line whose equations are

$$x + y = kz, \quad x - y = \frac{z}{k}.$$

2. Find the equation of the ruled surface generated by the line

$$\frac{x}{1} + \frac{y}{2} + \frac{z}{k} = 1, \; \frac{x}{2} + \frac{y}{1} + \frac{z}{m} = 1,$$

(a) when $\frac{k}{m} = 2$, (b) when $k + m = 1$, (c) when $km = 3$, and (d) when the perpendiculars from the origin upon the two planes are in the ratio 1 : 2.

184. Equation of generator. It frequently happens that the equation of a surface indicates at once that it is a ruled surface. For example, the equation $(x + y)^2 + (x + y)z - 3 = 0$ can be written

$$(x + y)(x + y + z) = 3.$$

Hence the straight line $x + y = \frac{3}{k}$, $x + y + z = k$ lies wholly on the surface, whatever value is given to k. This line is a generator for any value of k.

EXERCISES

1. Show that the hyperboloid of one sheet is a ruled surface.

SUGGESTION. The equation of the surface can be written

$$\left(\frac{y}{b} + \frac{z}{c}\right)\left(\frac{y}{b} - \frac{z}{c}\right) = \left(1 - \frac{x}{a}\right)\left(1 + \frac{x}{a}\right).$$

Hence the system of straight lines $\left(\frac{y}{b} + \frac{z}{c}\right) = k\left(1 + \frac{x}{a}\right)$, $\left(\frac{y}{b} - \frac{z}{c}\right) = \frac{1}{k}\left(1 - \frac{x}{a}\right)$ lies wholly on the surface.

2. Show that a second system of straight lines lies wholly on the hyperboloid of one sheet.

3. Show that the hyperbolic paraboloid is a ruled surface, having two systems of straight lines lying upon it.

4. Show that the three other nondegenerate conicoids are not ruled surfaces.

5. Show, by the method of this section, that the cone $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=0$ is a ruled surface.

6. Prove that $y^2-4yz+4z^2+xy-2xz=5$ is a ruled surface. Are there two systems of lines lying upon it? What is the form of the surface? How do the generators lie with respect to each other?

185. Tangent lines and planes. When two of the points in which a straight line meets a surface coincide at a point P, the line is called a **tangent line** to the surface and P is called its **point of contact.**

In general, all the tangent lines at P lie in a plane called the **tangent plane.** P is the **point of contact** of the plane.

To find the equation of the tangent plane at a given point on a surface, consider the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$

Let $P_1\equiv(x_1, y_1, z_1)$ and $P_2\equiv(x_2, y_2, x_2)$ be any two points in space. The equations of the line P_1P_2 are, in parametric form (Art. 168),

$$x=\frac{x_1+rx_2}{r+1},\ y=\frac{y_1+ry_2}{r+1},\ z=\frac{z_1+rz_2}{r+1}. \tag{1}$$

To find the coördinates of the points in which this line pierces the ellipsoid, substitute the values of x, y, and z in the equation of the surface and solve for r. These values of r, when placed in equations (1), give the coördinates sought.

The equation for r is readily found to be

$$r^2\left(\frac{x_2^2}{a^2}+\frac{y_2^2}{b^2}+\frac{z_2^2}{c^2}-1\right)+2r\left(\frac{x_1x_2}{a^2}+\frac{y_1y_2}{b^2}+\frac{z_1z_2}{c^2}-1\right)$$
$$+\left(\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}+\frac{z_1^2}{c^2}-1\right)=0. \tag{2}$$

Now suppose the point $P_1\equiv(x_1, y_1, z_1)$ lies on the ellipsoid. The absolute term in (2) is then zero, and one of the roots is $r_1=0$, as it should be. The other root is

$$r_2=\frac{-2\left(\frac{x_1x_2}{a^2}+\frac{y_1y_2}{b^2}+\frac{z_1z_2}{c^2}-1\right)}{\frac{x_2^2}{a^2}+\frac{y_2^2}{b^2}+\frac{z_2^2}{c^2}-1}.$$

In order that this root should be zero also, and therefore the two points of intersection coincide at P_1, it is necessary and sufficient that the numerator should vanish. Hence, when the point P_2 lies on the plane

$$\frac{x_1x}{a^2}+\frac{y_1y}{b^2}+\frac{z_1z}{c^2}-1=0, \tag{3}$$

the line P_1P_2 will touch the surface at P_1; and therefore (3) is the *equation of the tangent plane at* P_1.

When P_1 does not lie on the surface, equation (3) represents a plane called the **polar plane** of P_1 with respect to the ellipsoid.

A line perpendicular to the tangent plane at the point of contact is called the **normal** to the surface at this point.

EXERCISES

1. Show that the point $\left(1,\ 2,\ 2\frac{\sqrt{11}}{3}\right)$ lies on the ellipsoid $\frac{x^2}{4}+\frac{y^2}{9}+\frac{z^2}{16}=1.$ Find the equation of the tangent plane at this point, and the equations of the normal.

2. Derive the equation of a tangent plane to the hyperboloid of one sheet.

3. Derive the equation of a tangent plane to the hyperbolic paraboloid.

4. Derive the equation of a tangent plane to the quadric cone and show that any tangent plane passes through the vertex.

5. Show, by means of equation (2), Art. 185, that when P_2 lies on the polar plane of P_1, the segment P_1P_2 is divided externally and internally in the same ratio by the points where the line P_1P_2 meets the surface.

6. Show that the length of a tangent line to a sphere from the point (x_1, y_1, z_1) is equal to the square root of the result of substituting x_1, y_1, z_1 for x, y, z in the left member of the equation of the sphere, the right member being zero.

7. Show that the locus of points from which tangents of equal length may be drawn to two spheres is a plane. This plane is called the *radical plane* of the two spheres.

8. Prove that the radical planes of three spheres meet in a line called the radical axis of the three spheres.

9. Prove that the radical axis of three spheres is perpendicular to the plane of their centers.

10. Show by definition that the tangent plane to a ruled quadric contains the two generators which pass through the point of contact.

186. Circular sections. Certain planes cut the conicoids in circles. For example, consider the ellipsoid

$$\frac{x^2}{4}+\frac{y^2}{9}+\frac{z^2}{1}-1=0.$$

The coördinates of all points on the curve of intersection of this ellipsoid with the sphere $x^2+y^2+z^2-4=0$ will satisfy the equation

$$\left(\frac{x^2}{4}+\frac{y^2}{9}+\frac{z^2}{1}-1\right)-k(x^2+y^2+z^2-4)=0$$

whatever value is given to k. When $k=\frac{1}{4}$, the equation becomes

$$\frac{3\,z^2}{4}-\frac{5\,y^2}{36}=0 \tag{1}$$

and therefore represents two planes. Each plane cuts the sphere, and therefore the ellipsoid, in a circle.

Any plane parallel to either of the planes (1) cuts the ellipsoid in a circle. For, let

$$z\frac{\sqrt{3}}{2}+y\frac{\sqrt{5}}{6}-k=0 \text{ and } z\frac{\sqrt{3}}{2}-y\frac{\sqrt{5}}{6}-m=0 \tag{2}$$

be the equations of any two planes parallel to the two planes (1).

Combining the product of the equations (2) with the equation of the ellipsoid, it is easily seen that all points common to the planes (2) and the ellipsoid must satisfy the equation

$$\frac{x^2}{4}+\frac{y^2}{4}+\frac{z^2}{4}+k\left(z\frac{\sqrt{3}}{2}-y\frac{\sqrt{5}}{6}\right)+m\left(z\frac{\sqrt{3}}{2}+y\frac{\sqrt{5}}{6}\right)-km-1=0.$$

But this is the equation of a sphere and hence the planes (2) meet the ellipsoid in circles. There are thus two systems of parallel circular sections, each being parallel to one of the planes (1).

EXERCISES

1. Find the equations of the planes which cut circles from the ellipsoid $9\,x^2+25\,y^2+169\,z^2=1$.

2. For what values of k and m will the equations (2) be the equations of tangent planes to the ellipsoid?

3. Find the equations of the system of planes which cut the hyperboloid of one sheet $\frac{x^2}{9}+\frac{y^2}{25}-\frac{z^2}{169}=1$ in circles.

187. Asymptotic cones. Consider the hyperboloid of one sheet

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1.$$

Let $P_1 \equiv (x_1, y_1, z_1)$ be any point in space. The equations of the line joining the origin to P_1 are

$$x=rx_1,\ y=ry_1,\ z=rz_1,$$

r being a parameter; and the coördinates of the points in which this line meets the hyperboloid are found by substituting these values of x, y, and z in the equation of the surface and solving the resulting equation for r. We thus obtain for r the equation

$$r^2=\frac{1}{\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-\frac{z_1^2}{c^2}}.$$

It follows, from this equation, that as P_1 is made to approach the cone

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=0, \qquad (1)$$

the values of r increase in absolute value, becoming infinite when P_1 lies on the cone. Therefore no generator of the cone (1) ever meets the hyperboloid. Moreover, the sections of the cone and the hyperboloid, parallel to the XY-plane, approach coincidence as the cutting plane recedes from the XY-plane. The cone (1) is called the **asmyptotic cone.**

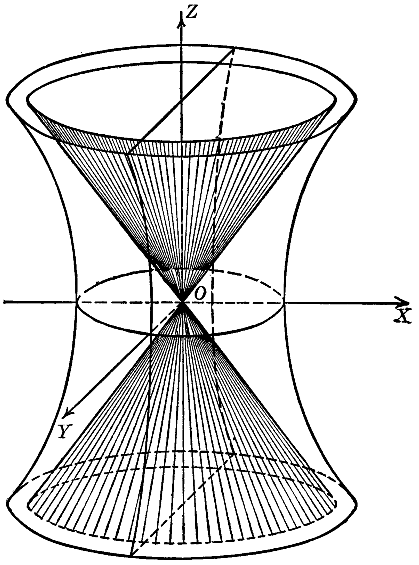

Fig. 138

EXERCISES

1. Find the equation of the asymptotic cone of the hyperboloid of two sheets.

2. Show that the asymptotic cone of the hyperbolic paraboloid consists of two planes.

3. Show that a plane determined by any generator of the hyperboloid of one sheet and the center is tangent to the asymptotic cone.

4. Show that neither the ellipsoid nor the elliptic paraboloid has an asymptotic cone.

188. Projecting cylinders of a curve in space. The cylinders whose generators intersect a given space-curve and are perpendicular to one of the coördinate planes are called the **projecting cylinders** of the curve.

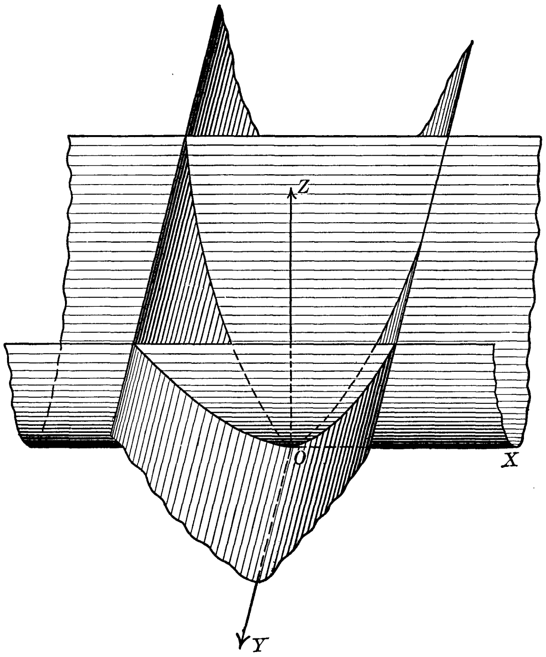

FIG. 139

To find the equations of the projecting cylinders, eliminate x, y, and z in turn from the equations of the curve. For example, consider the curve whose equations are

$$x^2 + y^2 = 8z, \quad x^2 - y^2 = 4z.$$

Eliminating x, y, and z in turn from these equations, the three projecting cylinders are found to be

$$y^2 = 2z, \quad x^2 = 6z, \quad \text{and} \quad x^2 - 3y^2 = 0.$$

The first two are parabolic cylinders, shown in the figure. They intersect in the given curve. The third equation decomposes into the two planes

$$x + \sqrt{3}\,y = 0 \quad \text{and} \quad x - \sqrt{3}\,y = 0$$

and shows that the given curve consists of two parabolas lying in these planes.

EXERCISES

1. Construct the following curves:

(a) $x^2 + y^2 = 25,\ y + z = 0.$ (b) $x^2 + y^2 - 4x = 0,\ x + y + z = 3.$

(c) $x^2 - y^2 = 4,\ x + y + z = 0.$

2. Find the equations of the projecting cylinders of the following curves:

(a) $x^2 + y^2 - 2y = 0,\ y^2 + z^2 = 4.$

(b) $2y^2 + z^2 + 4x - 4z = 0,\ y^2 + 3z^2 - 8x = 12z.$

(c) $x^2 + y^2 + z^2 = 25,\ x^2 + 4y^2 - z^2 = 0.$

The last is a *spherical conic.*

3. A point moves so as to be constantly 2 units from the Z-axis and 2 units from the point $(2, 0, 0)$. Find the equations of its locus and plot the curve.

189. Parametric equations of curves in space. If the coördinates of a point in space are each functions of a parameter, the locus of the point is a line in space, straight or curved. For example, the equations in Art. 168 are the parametric equations of a straight line in space. Again, the equations

$$x = r^3, \quad y = r^2, \quad z = r$$

are the parametric equations of a curve in space. The equations of the projecting cylinders of this curve are found by eliminating r from each pair of equations. Thus, the projecting cylinders are

$$x^2 = y^3, \quad x = z^3, \quad \text{and} \quad y = z^2.$$

190. The circular helix. An important curve in mechanics is the circular helix. Its parametric equations are

$$x = a\cos\theta,\ y = a\sin\theta,\ z = b\theta,$$

where θ is the parameter.

The equations of the projecting cylinders are

$$x^2 + y^2 = a^2,\ x = a\cos\frac{z}{b},\ y = a\sin\frac{z}{b}\cdot$$

Hence the curve lies on the right circular cylinder $x^2 + y^2 = a^2$. If b is a positive number, the XZ-cylinder stands on the curve $x = a\cos\frac{z}{b}$ or $ABCDE$; and the YZ-cylinder stands on the curve $y = a\sin\frac{z}{b}\cdot$ The helix is therefore a curve wound around the circular cylinder, the distance between two consecutive turns being $2b\pi$ (Fig. 140).

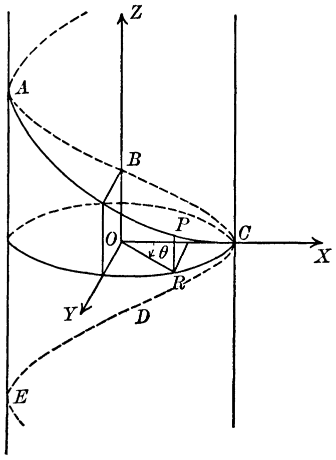

FIG. 140

EXERCISES

1. Plot the curves:

(a) $x = 2r,\ y = r^2,\ z = -\frac{r^4}{4}\cdot$ (b) $x = 6\cos\theta,\ y = 6\sin\theta,\ z = \frac{1 + \cos^2\theta}{4}$.

2. Construct a circular helix when b is a negative number.

3. Show that the equations

$$x = a\cos\theta,\ y = b\sin\theta,\ z = m\theta$$

are the equations of a helix wound on an elliptic cylinder.

4. Construct the curve

$$x = a\sec\theta,\ y = b\tan\theta,\ z = m\theta.$$

5. The three equations

$$\begin{aligned} kx + 2y + kz - 2k &= 0,\\ x + y + kz - k &= 0,\\ kx - y - zk - 1 &= 0, \end{aligned}$$

are the equations of three coaxial pencils of planes (Art. 152). Express the coördinates of the point of intersection of the three planes, for any value of k, in terms of k and thus show that the three pencils generate a space-curve. Construct the curve.

6. Show that the point (2, 1, 5) lies on the curve

$$x^2 + y^2 + z^2 = 30,\ y^2 = \frac{x}{2},$$

and find: (a) the equations of the tangent line to this curve at the given point, (b) the equation of the normal plane (perpendicular to the tangent line) at the given point.

ANSWERS

Art. 3. Page 10.

1. BA, AB, OB; 8, 8, 3; No. **2.** 73° on the Fahrenheit scale.
4. 205°, 170° **5.** 310°. **6.** 115°.

Art. 7. Page 13.

4. $P_1P_2 = 5$, $P_2P_3 = 3$, $P_1P_3 = 4$.
5. $P_1P_2 = \sqrt{37}$, $P_2P_3 = 3$, $P_1P_3 = 4$.
6. A square, one side 4 units, diagonal $4\sqrt{2}$, area 16.
7. (2, 0); each $\sqrt{2}$; each 2; 1.

Art. 9. Pages 15–16.

1. $(2.598, -1.5)$; $(2, -3.464)$; $(-1.25, 2.73)$.
2. $(\sqrt{58}, -66^\circ.8)$; $(2\sqrt{5}, 26^\circ.6)$; $(-\sqrt{34}, 59^\circ.03)$.
4. $y = \pm 3$, $\theta = \pm 36^\circ.8$.
5. $(-17\tfrac{7}{9}, 0)$.
9. $A \equiv (-8.66, 5)$ or $(10, 60^\circ)$. $B \equiv (0, 15)$ or $(15, 0^\circ)$. $C \equiv (10.39, 6)$ or $(12, -60^\circ)$.

Art. 11. Pages 19–20.

1. (*a*) 5 and 2; slope, $\frac{2}{5}$. (*b*) 1 and -9; slope, -9.
(*c*) 2 and 8; slope, 4. (*d*) 1 and -5; slope, -5.
2. 19° 6′; 40° 54′.
3. (*b*) 26° 34′; 75° 58′; 146° 19′.

Art. 12. Page 21.

1. 13.86. **2.** $\sqrt{13}$. **3.** 5.97.
4. 12.73; 2.23; 14.92. **5.** $(80, \frac{\pi}{4})$, 144 miles.

Art. 15. Page 24.

1. 38° 27′. **2.** 11.402, 7.616, 7.211; 100° 30′, 41° 3′, 38° 27 .
4. 1.792. **5.** 10. **6.** 35° 24′.

Art. 17. Pages 25–26.

1. (0, 3.5); (1, – 1.5); (3, 0).

5. (0, 2) and (3, 3).

6. (1.125, .25).

7. (1, 0) and (0, – 2).

Art. 19. Page 28.

1. – 6. **3.** – 7.5. **4.** 6.897. **6.** 2.5.

Art. 20. Page 30.

1. 88.5. **3.** 22.935. **4.** 151.

Art. 21. Pages 31–32.

1. 122.5. **2.** 25 acres and $\frac{1}{2}$ a sq. rd. **3.** 60,294 sq. ft.

4. (*a*)
$$\begin{cases} x = -94.2 & -27.8 & 188.0 & 54.1 \\ y = 66.0 & 157.3 & 68.3 & -166.3 \end{cases}$$

(*b*) N. 36° E.; S. 67° 36′ E.; S. 29° 46′ W.; N. 26° 48′ W.; N. 1° 12′ E.; N. 50° 45′ W.

(*c*) 42,277.06 sq. ft.

5. 150.5. **6.** 300, slope, $\frac{2}{3}$.

Art. 30. Page 39.

1. Line of symmetry, $x = 1$; intercepts on X-axis, -1 and 3; intercept on Y-axis, -3; turning point $(1, -4)$.

2. Line of symmetry, $y = 1$; intercept on X-axis, $\frac{1}{2}$; intercept on Y-axis, 1.

3. $(\pm 2, \pm 2)$.

6. $\left(-\dfrac{b}{2a}, \dfrac{4ac - b^2}{4a}\right)$.

7. Area $= 4x\sqrt{25 - x^2}$, where x is one half the length of one side. Turning point for $x = \dfrac{5\sqrt{2}}{2}$. Max. rect. is a square whose side is $5\sqrt{2}$.

8. Number of sq. ft. of lumber is $\dfrac{432}{x} + x^2$, where x is the length of the side of the base. Height of the box requiring the least amount of lumber is 3 feet.

Art. 34. Page 42.

3. (*a*) $y = a$. (*b*) $(x^2 + y^2 + 2ax)^2 = 4a^2(x^2 + y^2)$.
(*c*) $(x^2 + y^2 - ax)^2 = a^2(x^2 + y^2)$.

4. $r = 2\cos\theta$.

Art. 41. Page 50.

9. $pv = 4$. **10.** $l = \dfrac{t}{50000} + 1$.

11. $i = \dfrac{(93000000)^2}{d^2}$. $i = 1.93$, nearly.

Art. 45. Page 56.

1. (*a*) $x^2+y^2-2y-8=0.$ (*b*) $x^2+y^2+4x=0.$
(*c*) $x^2+y^2+8x-6y+16=0.$ (*d*) $x^2+y^2-2x-4y-31=0.$
2. $x^2+y^2-4x-6y=0.$ **3.** $x^2+y^2+4x-6y-13=0.$
4. $x^2+y^2-5x+\frac{1}{2}y-\frac{3}{2}=0.$ **5.** $10x-8y+3=0.$
6. $2(a-c)x+2(b-d)y-b^2+d^2-a^2+c^2=0.$
7. $x^2+y^2=16.$ **8.** $x^2+y^2=25.$
9. $x^2+y^2-6x+4y-12=0.$

Art. 46. Page 57.

1. (*a*) $(3, 0)$; 5. (*b*) $(3, -2)$; $3\sqrt{2}$. (*c*) $(\frac{5}{3}, 4)$; $\frac{13}{3}$. (*d*) $(-1, 2)$; 0.
(*e*) $(4, 0)$; 4. (*f*) $(\frac{2}{7}, \frac{1}{14})$; $\frac{\sqrt{101}}{14}$. (*g*) imag. (*h*) imag.
2. $(19, \frac{32}{3})$; 19.565, nearly.

Art. 48. Page 59.

1. (*a*) $x-y+1=0.$ (*b*) $3x+2y+1=0.$ (*c*) $x+9y+13=0.$
(*d*) $7x+4y-5=0.$
5. (*a*) -1; $\frac{9}{2}, \frac{9}{2}$. (*b*) $\frac{2}{3}$; $\frac{3}{2}, -1$. (*c*) 3; $-\frac{2}{3}$, 2. (*d*) $-\frac{5}{3}$; $\frac{4}{5}, \frac{4}{3}$.

Art. 50. Pages 62–63.

1. $\frac{x^2}{16}+\frac{y^2}{7}=1$; $e=\frac{3}{4}$.
2. ± 4, 0.
3. (*a*) $\frac{x^2}{25}+\frac{y^2}{9}=1.$ (*b*) $\frac{x^2}{36}+\frac{y^2}{9}=1.$ (*c*) $\frac{x^2}{25}+\frac{4y^2}{25}=1.$ (*d*) $\frac{x^2}{100}+\frac{y^2}{50}=1.$
(*e*) $\frac{x^2}{50}+\frac{y^2}{25}=1.$
4. $8\frac{7}{16}$, $7\frac{9}{16}$.
5. (*a*) $a=\sqrt{2},\ b=\sqrt{3},\ c=\frac{1}{3}\sqrt{3}.$ (*b*) $a=\sqrt{3},\ b=\sqrt{2},\ c=\frac{1}{3}\sqrt{3}.$
(*c*) $a=\sqrt{2},\ b=\frac{1}{3}\sqrt{6},\ c=\sqrt{\frac{2}{3}}.$ (*d*) $a=1,\ b=\frac{1}{2}\sqrt{2},\ c=\frac{1}{2}\sqrt{2}.$
6. (*a*) $2\sqrt{\frac{4}{3}}$, (*b*) $\frac{4\sqrt{3}}{3}$, (*c*) $\frac{2}{3}\sqrt{2}$, (*d*) 1.

Art. 52. Pages 65–66.

1. $\frac{x^2}{9}-\frac{y^2}{7}=1.$ **2.** $\frac{x^2}{36}-\frac{y^2}{28}=1.$ **3.** 7.03+ and 1.03+.
4. (*a*) 3, 2; $\frac{1}{3}\sqrt{13}$. (*b*) 2, 3; $\frac{1}{2}\sqrt{13}$. (*c*) 1, 4; $\sqrt{17}$. (*d*) $2\sqrt{m}$, $\sqrt{m}$; $\frac{1}{2}\sqrt{5}$.
5. (*a*) $\sqrt{6}$, 2; $\frac{1}{2}\sqrt{10}$. (*b*) 4, 2; $\sqrt{5}$. (*c*) $\sqrt{\frac{n}{m}}$, $\sqrt{n}$; $\sqrt{\frac{1+m}{m}}$.
6. (4 *a*) $2\frac{2}{3}$. (4 *b*) 9. (4 *c*) 32. (4 *d*) $\sqrt{m}$. (5 *a*) 6. (5 *b*) 16. (5 *c*) $\frac{2\sqrt{n}}{m}$.

Art. 53. Page 67.

1. $(1, 0), 2.$ **2.** $y^2 = 8(x-1).$ **3.** $x^2 = 4(y-1).$

4. $(a)\ y^2 = 8x.\ (b)\ x^2 = 8y.$ **6.** $(a)\ 8.\ (b)\ 4.\ (c)\ 6.\ (d)\ 10.$

Art. 57. Pages 70–71.

1. $r^2 - 6r\cos(\theta - 60°) = 7; \quad \sqrt{37}.$

2. $r\cos(\theta - 60°) = 5; \quad x + y\sqrt{3} = 10.$

3. $r = 8\sin\theta; \quad x^2 + y^2 = 8y.$

6. $r = \pm 2a\cos(\theta - 45°); \quad x^2 + y^2 \pm ax\sqrt{2} \pm ay\sqrt{2} = 0.$

Art. 59. Pages 72–73.

1. $r = \dfrac{7}{4 - 3\cos\theta}.$ **2.** $r = \dfrac{5}{3\cos\theta - 2}.$ **5.** $r = \dfrac{6}{1 - \cos\theta}.$

9. $r = \dfrac{c^2 - a^2}{a - c\cos\theta}.$ **10.** $r = \dfrac{a^2 - c^2}{a + c\cos\theta}.$ **12.** $r = \dfrac{a}{\sqrt{2}\cos\theta - 1}.$

Art. 62. Page 76.

1. $(a)\ a = 3,\ b = 2,\ c = \sqrt{5},\ e = \dfrac{\sqrt{5}}{3}.$ $(b)\ a = \dfrac{\sqrt{105}}{7},\ b = \dfrac{\sqrt{165}}{11},$ $c = \sqrt{\tfrac{60}{77}},\ e = \sqrt{\tfrac{4}{11}}.$ $(c)\ a = 10,\ b = 5,\ c = 5\sqrt{5},\ e = \tfrac{1}{2}\sqrt{5}.$ $(d)\ a = 2\sqrt{\tfrac{29}{17}},$ $b = \dfrac{\sqrt{116}}{5},\ c = \tfrac{2}{5}\sqrt{\tfrac{1218}{17}},\ e = \sqrt{\tfrac{42}{17}}.$ $(e)\ a = 8,\ b = 5,\ c = \sqrt{39},\ e = \dfrac{\sqrt{39}}{8}.$ $(f)\ a = 8,\ b = 5,\ c = \sqrt{89},\ e = \dfrac{\sqrt{89}}{8}.$

2. $2y + x = 0$ and $2y - x = 0.$ **3.** $5\tfrac{1}{3}.$

4. $r = \dfrac{16}{5\cos\theta - 3},\ \theta = \pm \text{arc tan } \tfrac{4}{3}.$

Art. 72. Page 87.

3. $a = 2,\ T = \tfrac{1}{200}.$ $x = 2\cos\phi + \sqrt{c^2 - 4\sin^2\phi} - c$, where c is the length of the connecting rod.

Art. 78. Page 94.

1. $3x' + 7y' = 10.$ **3.** $x'y' = -5.$ **4.** $y'^2 = a\sqrt{2}\,x'.$

Art. 81. Page 97.

2. $\left(3\tfrac{1}{2},\ 5 \pm \dfrac{3\sqrt{3}}{2}\right).$ **7.** $x = 4 - t^2,\ y = t(4 - t^2).$

3. $(4\tfrac{3}{7},\ -3\tfrac{1}{7}).$ **8.** $x = 2\left(\dfrac{t^2 - 1}{t^2 + 1}\right),\ y = 2t\left(\dfrac{t^2 - 1}{t^2 + 1}\right).$

6. $y = \pm\dfrac{\sqrt{7}}{3}x.$ **9.** $x = a\cos^4\theta,\ y = a\sin^4\theta.$

10. $x = r\sqrt{1 - \dfrac{(a+b)^2\sin^2\theta}{r^2}} + b\cos\theta,\ y = a\sin\theta.$

Art. 82. Pages 98–99.

2. (1) $\frac{x}{2} - \frac{y}{3} = 1.$ (2) $y = 4(x + 1).$ (3) $y = -2x + 3.$
(4) $y + 5x + 10 = 0.$ (5) $3x + 2y - 6 = 0.$ (6) $x + y = 1.$
(7) $c^2y + ACx = AB.$

Art. 83. Page 100.

2. $-10x - 8y + 40 = 0.$ $-\frac{1}{5}x - \frac{4}{25}y + \frac{4}{5} = 0.$
3. $\frac{9}{4}x - 3y + 9 = 0.$ $\frac{3}{2}x - 2y + 6 + 0.$
4. 10, -4, -6 or -15, 6, 9.

Art. 84. Page 101.

1. $11x - 35y = 0.$
2. $15x - 18y - 320 = 0$, $4x + 5y - 3 = 0$, $49x + 98y - 272 = 0.$
3. $3x + 2y + 7 = 0.$ $5x - y + 8 = 0.$
4. $x\sqrt{3} - y - (\sqrt{3} - 3) = 0.$ **5.** $3y - 10x - 4 = 0.$

Art. 87. Page 104.

1. (*a*) $x + y\sqrt{3} - 10 = 0.$ (*b*) $x - y\sqrt{3} + 10 = 0.$ (*c*) $x\sqrt{3} - y + 10 = 0.$
(*d*) $x + y = 0.$ (*e*) $x + y = \frac{1}{2}\sqrt{2}.$ (*f*) $x - y\sqrt{3} - 12 = 0.$
4. $y = 5$ and $21x + 20y - 145 = 0.$

Art. 88. Page 105.

1. $\frac{3\sqrt{13}}{13}.$ **2.** $2.35^{+}.$ **3.** $x^2 + y^2 = \frac{625}{169}.$
4. $\frac{13}{2}$, $4x^2 + 4y^2 + 24x - 32y - 69 = 0.$
5. (*a*) $x^2 + y^2 = 5.$ (*b*) $x^2 + y^2 + 8x - 3y + 5.025 = 0.$
(*c*) $x^2 + y^2 + (6 - \sqrt{2})x - (4 + \sqrt{2})y + \frac{25}{2} = 0.$
(*d*) $x^2 + y^2 + 3(2 - \sqrt{2})x + 3(2 - \sqrt{2})y + \frac{9(3 - 2\sqrt{2})}{2} = 0.$

Art. 89. Page 106.

1. 45°. **2.** 4° 46′, nearly. **3.** 105° 15′, 63° 26′, 11° 19′.
5. $y = -\frac{5}{2}$, $x - y = \frac{60}{17}$, $17x + 7y = 0.$
8. (*a*) $2x - 3y = 6.$ (*b*) $3x \pm 4y = 15.$ (*c*) $3y - 2x = 5\sqrt{13}.$

Art. 90. Page 107.

1. (1) $\frac{x^2}{9} + \frac{y^2}{4} = 1.$ (2) $\frac{x^2}{12} + \frac{y^2}{9} = 1.$ (3) $\frac{x^2}{36} + \frac{y^2}{20} = 1.$
(4) $\frac{x^2}{25} + \frac{y^2}{16} = 1.$ (5) $\frac{x^2}{25} + \frac{y^2}{16} = 1.$ (6) $\frac{x^2}{144} + \frac{y^2}{128} = 1.$

2. (1) $a = 5,\ b = 1,\ c = \sqrt{26},\ e = \dfrac{\sqrt{26}}{5}.$

(2) $a = 5,\ b = 1,\ c = \sqrt{24},\ e = \dfrac{\sqrt{24}}{5}.$

(3) $a = 2,\ b = 3,\ c = \sqrt{13},\ e = \dfrac{\sqrt{13}}{2}.$

(4) $a = 2,\ b = 3,\ c = \sqrt{5},\ e = \dfrac{\sqrt{5}}{3}.$

(5) $a = \sqrt{10},\ b = 2,\ c = \sqrt{14},\ e = \sqrt{\tfrac{7}{5}}.$

(6) $a = \sqrt{10},\ b = 2,\ c = \sqrt{6},\ e = \sqrt{\tfrac{3}{5}}.$

3. $\dfrac{5\sqrt{2}}{3}$ and $\dfrac{13\sqrt{2}}{3}.$ **4.** $\dfrac{\sqrt{13}}{6}(9 + 2\sqrt{5})$ and $\dfrac{\sqrt{13}}{6}(9 - 2\sqrt{5}).$

Art. 94. Pages 110–111.

1. $\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1.\quad r = \dfrac{4}{3 - \sqrt{5}\cos\theta}.\quad x = 3\cos t,\ y = 2\sin t.$

2. $\dfrac{x^2}{9} - \dfrac{4y^2}{45} = 1.\quad r = \dfrac{15}{6\cos\theta - 4}.\quad x = 3\sec t,\ y = \dfrac{3\sqrt{5}}{2}\tan t.$

3. $\dfrac{x^2}{36} + \dfrac{y^2}{27} = 1.\quad r = \dfrac{27}{6 - 3\cos\theta}.\quad x = 6\cos t,\ y = 3\sqrt{3}\sin t.$

4. $\dfrac{x^2}{16} - \dfrac{y^2}{25} = 1.\quad r = \dfrac{25}{\sqrt{41}\cos\theta - 4}.\quad x = 4\sec t,\ y = 5\tan t.$

5. $c = 4,\ \dfrac{a}{e} = \dfrac{25}{4}.$

Art. 95. Pages 113–115.

1. (*a*) $y = \tfrac{1}{2}x + 2.$ (*b*) $y = -\tfrac{4}{3}x \pm \tfrac{20}{3}.$ (*c*) $y = -\tfrac{1}{4}x \pm \sqrt{10}.$

(*d*) $y - 4 = \dfrac{-4 \pm \sqrt{91}}{9}(x - 3).$ (*e*) $y = -\tfrac{2}{3}x \pm 5.$

2. (*a*) $y = \pm\tfrac{3}{4}x \pm \tfrac{5}{3}.$ (*b*) $y = \pm x \pm 5.$ (*c*) $y = \pm 6x \pm 7\sqrt{37}.$

Art. 96. Page 116.

3. (*a*) $(4, 4).$ (*b*) $(\pm\tfrac{16}{5}, \pm\tfrac{12}{5}).$ (*c*) $(\pm\tfrac{2}{5}\sqrt{10}, \pm\tfrac{9}{10}\sqrt{10}).$

(*d*) $\left(\dfrac{12(-9 \mp 4\sqrt{91})}{55}, \dfrac{-9(16 \pm \sqrt{91})}{55}\right).$ (*e*) $(\pm\tfrac{24}{5}, \pm\tfrac{9}{5}).$

4. (*a*) $\left\{\begin{matrix}\text{Parabola, } (\tfrac{20}{9}, \pm\tfrac{10}{3}) \\ \text{Circle, } (-\tfrac{4}{5}, \pm\tfrac{16}{15})\end{matrix}\right\}.$ (*b*) $\left\{\begin{matrix}\text{Ellipse, } (\pm\tfrac{16}{5}, \pm\tfrac{9}{5}) \\ \text{Hyperbola, } (\pm\tfrac{32}{5}, \pm\tfrac{7}{5})\end{matrix}\right\}.$

(*c*) $\left\{\begin{matrix}\text{Circle, } \left(\pm\dfrac{42\sqrt{37}}{37}, \pm\dfrac{7\sqrt{37}}{37}\right) \\ \text{Ellipse, } \left(\pm\dfrac{300\sqrt{37}}{259}, \pm\dfrac{13\sqrt{37}}{259}\right)\end{matrix}\right\}.$

Art. 97. Page 118.

1. $3x + 8y = 19.$ **3.** $3x + y = 7.$ **4.** $y = 4,\ y = -\frac{3}{2}x + \frac{17}{2}.$
5. $x + 2y + 6 = 0.$ **6.** 108° 26′.
10. (*a*) $y = 3x - 4.$ (*b*) $11x - y - 18 = 0.$ (*c*) $3y = x + 4.$
(*d*) $11x - 2y\sqrt{6} - 21 = 0.$

Art. 99. Page 120.

1. $x_1y - y_1x = 0.$ **2.** $\frac{y - y_1}{x - x_1} = -\frac{y_1}{2p};\ \frac{y - y_1}{x - x_1} = -\frac{a^2y_1}{b^2x_1}.$
3. $\frac{2}{3}\sqrt{73},\ \frac{1}{4}\sqrt{73},\ 5\frac{1}{3},\ \frac{3}{4}.$ **4.** $27y + 18x - 88 = 0.$ **7.** $8x - 3y = 18.$
8. $m = \frac{\sqrt{65}}{3}$; subtangent $= \frac{192\sqrt{65}}{65}$; subnormal $= \frac{4\sqrt{65}}{3}.$

Art. 102. Pages 124–125.

2. $9y - 2x\sqrt{3} = 0,\quad 3y + 2x\sqrt{3} = 0.$

4. $y_1x - x_1y = 0,\quad \frac{x_1x}{a^2} - \frac{y_1y}{b^2} = 0.$

Art. 103. Pages 126–127.

1. $y = \frac{1}{3}x.$ **2.** $y = -9.$ **3.** $x + 2y = 8.$

4. $13x + 11y = 46.$ **5.** $y = x - 1.$ **6.** $y = \frac{b^2}{a^2m}x.$

Art. 104. Pages 128–129.

1. (1) $x - 8y = 16.$ (2) $x + 2y + 6 = 0.$ (3) $5x + 8y + 12 = 0.$
(4) $5x - 6y = 5.$

2. $(15,\ -10).$ **3.** $(\frac{50}{7},\ \frac{20}{7}).$ **4.** $(-10,\ 4).$

5. $\left(-\frac{Aa^2}{C},\ -\frac{Bb^2}{C}\right).$ **6.** $\left(\frac{a^2b^2x_1}{a^2{y_1}^2 + b^2{x_1}^2},\ \frac{a^2b^2y_1}{a^2{y_1}^2 + b^2{x_1}^2}\right).$

Art. 109. Page 138.

1. $x^2 + 2y^2 = 10,\quad 5x^2 - 120y^2 = 24.$

Miscellaneous Exercises. Pages 138–140.

1. $(2 \pm 3\sqrt{6})x + 2(6 \mp \sqrt{6})y = 40.$

2. $\left(\frac{149 \mp 12\sqrt{15}}{2}\right)x - \left(\frac{-54\sqrt{5} \pm 24\sqrt{3}}{17}\right)y = 18.$

3. (1) $\left(\frac{4 \mp 6\sqrt{6}}{5},\ \frac{6 \pm \sqrt{6}}{5}\right).$ (2) $\left(\frac{149 \mp 12\sqrt{15}}{4},\ \frac{-18\sqrt{5} \pm 8\sqrt{3}}{17}\right).$

4. $\sqrt{15}$. **5.** $\frac{1}{2}\sqrt{15}$. **6.** $B^2 = AC$.

10. $x + y + p = 0$. **11.** 94,559,610 and 91,440,390.

12. $\text{arc tan}\,\dfrac{a^2m^2 + b^2}{m(a^2 - b^2)}$. **13.** 500,000.

15. $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = \dfrac{x}{a^2}$. **17.** $y = x(m + m')$.

Art. 111. Page 145.

1. (*a*) Ellipse, center $(\frac{3}{2}, -\frac{2}{3})$, foci $(\frac{3}{2} \pm \frac{1}{6}\sqrt{95}, -\frac{2}{3})$.
(*b*) Ellipse, center $(3, -\frac{1}{4})$, foci $(3 \pm \frac{3}{4}\sqrt{17}, -\frac{1}{4})$.
(*c*) Hyperbola, center $(\frac{1}{2}, 0)$, foci $(\frac{1}{2}, \pm \frac{7}{6}\sqrt{3})$.
(*d*) Parabola, vertex $(-2, -\frac{19}{2})$, focus $(-2, -9)$.
(*e*) Hyperbola, center $(0, 3)$, foci $(0, 3 \pm 2\sqrt{3})$.
(*f*) Parabola, vertex $(\frac{11}{2}, 2)$, focus $(5, 2)$.

2. (*a*) Circle. (*b*) Hyperbola. (*c*) Two lines parallel to Y-axis.
(*d*) One line parallel to X-axis. (*e*) Imaginary lines.
(*f*) Two lines parallel to X-axis.

Art. 112. Page 148.

1. Ellipse, 45°, $e = .577^+$.

2. (*a*) Ellipse. (*b*) Imag. ellipse. (*c*) Imag. lines.
(*d*) Hyperbola. (*c*) Real interesting lines.

3. (*a*) $3x^2 + y^2 = 2$. (*b*) $3x^2 - 7y^2 = 8$. (*c*) $7x^2 - 6y^2 = 14$.
(*d*) $2x^2 + y^2 = 2$. (*e*) $x^2 - 12y^2 = -9$.

Art. 114. Page 152.

1. (*a*) $(5x - y - 1)(x + y + 5) = 0$.

(*b*) $\dfrac{x^2}{2(\sqrt{2} - 1)} - \dfrac{y^2}{2(\sqrt{2} + 1)} = -1$; $\theta = 67° 30'$; center $(-1, 0)$.

(*c*) $\dfrac{x^2}{2(\sqrt{2} - 1)} - \dfrac{y^2}{2(\sqrt{2} + 1)} = -1$; $\theta = 67° 30'$; center $(-1, \frac{1}{2})$.

(*d*) $3x^2 + y^2 = 6$; $\theta = 45°$; center $(1, -2)$.

(*e*) $\dfrac{x^2}{3 + \sqrt{5}} + \dfrac{y^2}{3 - \sqrt{5}} = 1$; $\theta = 13° 18'$; center $(1, 1)$.

(*f*) Imaginary lines.

(*g*) $\dfrac{x^2}{3 + \sqrt{5}} + \dfrac{y^2}{3 - \sqrt{5}} = 0$; $\theta = \text{arc tan}\, 2$; center $(3, 3)$.

Art. 115. Page 154.

1. (*a*) $y^2 = -\frac{4\sqrt{2}}{3}x$; $\theta = 45°$; vertex $\left(-\frac{5\sqrt{2}}{3}, -\frac{2\sqrt{2}}{3}\right)$.

(*b*) $y^2 = -\frac{\sqrt{2}}{4}x$; $\theta = 45°$; vertex $\left(\frac{25\sqrt{2}}{4}, \frac{3\sqrt{2}}{8}\right)$.

(*c*) $x^2 = -\frac{4\sqrt{5}}{25}y$; $\theta = \text{arc sin}\,\frac{1}{5}\sqrt{5}$; vertex $\left(\frac{4\sqrt{5}}{25}, \frac{4\sqrt{5}}{25}\right)$.

(*d*) $x^2 = \frac{6\,y}{13\sqrt{13}}$; $\theta = \frac{1}{2}\,\text{arc tan}\,\frac{12}{5}$; vertex $\left(\frac{2}{13\sqrt{13}}, -\frac{2\sqrt{13}}{507}\right)$.

(*e*) $x^2 = -\frac{4\,y}{\sqrt{5}}$; $\theta = \text{arc sin}\,\frac{2\sqrt{5}}{5}$; vertex $\left(\frac{-\sqrt{5}}{5}, 0\right)$.

Art. 116. Page 155.

1. (*a*) $x - y - 1 = 0$. (*b*) Imaginary lines. (*c*) $x + y \pm 1 = 0$.
(*d*) $3\,x - 2\,y + 2 = 0$ and $3\,x - 2\,y + 3 = 0$.

Art. 117. Pages 156–157.

1. (*a*) Hyperbola. (*b*) Real ellipse. (*c*) Parabola. (*d*) Hyperbola.
2. (*a*) Parabola. (*b*) Two real lines. (*c*) Real ellipse.
(*d*) Two real lines.
4. $k = \frac{1}{2}$, imaginary.
7. Vertex $(\frac{8}{5}, \frac{4}{5})$; focus $(\frac{9}{5}, \frac{9}{10})$; directrix $4\,x + 2\,y = 7$.

Art. 118. Page 159.

1. (*a*) $3\,x + 8\,y - 5 = 0$ and $3\,x - 8\,y + 20 = 0$. (*b*) $x + y = 4$.
(*c*) $y + 4 = 0$. (*d*) $x + 2\,y + 3 = 0$. (*e*) $8\,x - 37\,y + 18 = 0$ and $3\,x + 13\,y - 18 = 0$.
2. $y - 4\,x = 8$.

Art. 119. Page 162.

1. $(5 \pm \sqrt{13})x - 2(7 \pm 2\sqrt{13})y - (13 \pm 5\sqrt{13}) = 0$.
2. $\begin{cases} \text{Axes, } x + y = 3 \text{ and } x - y = 9. \\ \text{Asymptotes, } (-15 \mp 7\sqrt{5})x \pm 2\,y\sqrt{5} + 15 \pm 2\sqrt{5} = 0. \end{cases}$
3. $y - x + 1 = 0$, $x + y - 3 = 0$, $x + y - 1 = 0$.

Art. 121. Pages 163–164.

1. $17\,x^2 + 105\,xy - 48\,y^2 + 210\,x + 39 = 0$.
2. (*a*) $5\,x + 5\,y + 2 = 0$. (*b*) $3\,x - 2\,y = 0$. (*c*) $6\,x + 1 = 0$.
(*d*) $4\,x - 2\,ay + a^2 - c^2 + d^2 = 4$.
4. $(\frac{1}{3}, -3)$.

Art. 122. Page 165.

1. $x^2 + 2xy + y^2 - x - 11 = 0$ and $x^2 - 2xy + y^2 - x - 7 = 0$.

2. $x^2 + 4xy + 4y^2 - x - 2 = 0$ and $x^2 - 4xy + 4y^2 - x - 2 = 0$.

Art. 123. Page 166.

1. (*a*) $x + \frac{3}{2} = \pm 2(y - \frac{7}{4})$, $3x + 1 = \pm(y - 14)$, $x + 15 = \pm(7y - 10)$.
(*b*) $7y + 36 = 0$, $x\sqrt{7} \pm 12 = 0$.
(*c*) $(2\sqrt{2} \pm \sqrt{5})x - (\sqrt{5} \mp \sqrt{2})y = \pm(4\sqrt{5} \pm 2\sqrt{2})$, $x - 2y = 0$, $x + y = 0$.

2. (*a*) $(-5, 0)$, $(-4, 3)$, $(3, 4)$, $(\frac{24}{5}, -\frac{7}{5})$. (*b*) $\left(\pm\frac{12\sqrt{7}}{7}, -\frac{36}{7}\right)$.
(*c*) $\left(\pm\frac{4\sqrt{10}}{5}, \pm\frac{2\sqrt{10}}{5}\right)$, $(\pm 2, \mp 2)$.

Art. 124. Page 167.

1. (*a*) $22x^2 - 30xy + 7y^2 - 22x + 9y = 0$.
(*b*) $24x^2 - 73xy + 29y^2 + 106x - 126y + 40 = 0$.

Miscellaneous Exercises. Pages 167–168.

2. $(2, -3)$.

4. $x^2 + y^2 - 18x - 36y + 81 = 0$ and $x^2 + y^2 - 2x - 4y + 1 = 0$.

5. $(2, 0)$ and $(5, 0)$. **6.** $(\frac{140}{13}, -\frac{85}{13})$.

7. (*a*) $y^2 = -\frac{4}{5}x$. (*b*) $\frac{x^2}{16} - \frac{y^2}{4} = -1$. (*c*) $\frac{x^2}{9} + \frac{y^2}{4} = 1$.
(*d*) $y^2 = \frac{10\sqrt{13}}{13}x$. (*e*) $\frac{x^2}{4(5 + \sqrt{5})} + \frac{y^2}{4(5 - \sqrt{5})} = 1$.
(*f*) $\frac{x^2}{2} - \frac{y^2}{3} = 1$. (*g*) $y^2 = \frac{3\sqrt{2}}{2}x$. (*h*) $y^2 = \frac{2\sqrt{5}}{5}x$.
(*i*) $(5x - 2y + 3)(5x - 2y - 2) = 0$. (*j*) $x - 3y - 1 = 0$.
(*k*) $x^2 - y^2 = -10\sqrt{2}$. (*l*) $x^2 - y^2 = -40$. (*m*) Imag. lines.
(*n*) $y^2 = -\frac{510}{2197}x$. (*o*) $(x - 2y - 2)(x + y + 1) = 0$.

Art. 145. Page 200.

2. Each, $\frac{5\sqrt{3}}{3}$. **3.** $(7, -4, -3)$. **4.** $(1, 0, 11)$, 7.

Art. 147. Page 201.

2. Lengths of the sides, $\sqrt{83}$, $\sqrt{217}$, $\sqrt{54}$.

4. Terminal point, $(\frac{1}{2}, \frac{8}{3}, -3 \pm \frac{5}{6}\sqrt{23})$.

6. Direction cosines, $\frac{4}{5}, \frac{3}{5}, 0$; $\frac{2}{3}, -\frac{1}{3}, -\frac{2}{3}$. **8.** 60° or 120°.

Art. 151. Page 204.

1. (*a*) 90°. (*b*) arc cos $-\frac{4}{21}$. (*c*) arc cos $\frac{14}{39}$.

7. $4\frac{1}{3}$. **8.** $1\frac{1}{13}$.

Art. 152. Pages 205–206.

1. (*a*) $(5\frac{2}{3}, -2\frac{2}{3}, 3\frac{1}{3})$. (*b*) $(3, 1\frac{2}{5}, 3\frac{2}{5})$.

3. $(-1\frac{2}{3}, 2, 4\frac{2}{3})$ and $(-\frac{1}{6}, 0, 2\frac{1}{6})$. **8.** -2. $\frac{1}{2}$. 1.

Art. 155. Pages 208–209.

1. $x^2 + y^2 + z^2 - 10x + 4y - 6z + 37 = 0$.
$x^2 + y^2 + z^2 - 4x + 6y + 12z = 0$. $x^2 + y^2 + z^2 - 2az = 0$.

2. (*a*) $(1, -3, 4)$, $r = 2$. (*b*) $(-5, 2, -1)$, $r = 5$.
(*c*) $(-2, -2, -3)$, $r = 4$. (*d*) $(-3, 0, 0)$, $r = 3$. (*e*) Imaginary.

3. $x^2 + y^2 + z^2 - 2x - 8y - 16z = 0$. **4.** $x^2 + y^2 + z^2 - 4x = 217$.

5. $x^2 + y^2 + z^2 = 4$ and $x^2 + y^2 + z^2 = 576$.

Art. 156. Page 210.

1. $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{b^2} = 1$. **2.** $y^2 + z^2 = 4px$.

3. $\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{b^2} = 1$ and $\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{b^2} = -1$.

5. $y^2 + z^2 = 4px$. $y^4 = 16p^2(x^2 + z^2)$.

Art. 159. Page 212.

1. $x^2 + y^2 = z^2 \tan^2 \theta$. **2.** $\frac{x^2}{4} + \frac{y^2}{4} - \frac{(z-3)^2}{9} = 0$.

4. $(\frac{12}{13}, 0, \frac{18}{13})$ and $(-\frac{12}{65}, 0, \frac{138}{65})$. **5.** $bx = az$. About the Z-axis.

Art. 160. Pages 214–215.

1. (*a*) $x\sqrt{2} + y + z = 8$. (*b*) $x + y\sqrt{2} - z + 12 = 0$.
(*c*) $6x - 2y + 3z = 56$. (*d*) $2x + y + 2z + 15 = 0$.

2. (*a*) $3x - 2y + 6z = 49$. (*b*) $2x - 5y + z = 30$.
(*c*) $3x + 4y - 2z + 29$.

3. (*a*) $\frac{6}{7}x - \frac{3}{7}y + \frac{2}{7}z = 1$. (*b*) $-\frac{1}{2}x + \frac{\sqrt{2}}{2}y - \frac{1}{2}z = 4$.
(*c*) $\frac{1}{\sqrt{21}}x - \frac{4}{\sqrt{21}}y - \frac{2}{\sqrt{21}}z = \frac{3}{\sqrt{21}}$. (*d*) $\frac{1}{\sqrt{14}}x - \frac{2}{\sqrt{14}}y - \frac{3}{\sqrt{14}}z = 0$.

5. 54.

Art. 162. Pages 215–216.

2. $16x + 6y - z = 14$.

4. Area of XZ-proj. $= 4$. Area of YZ-proj. $= 6$. Area of XY-proj. $= 12$.

Art. 163. Page 217.

1. $3x + 2y + 6z = 12$. **2.** $x - 3y - 2z = 0$.

Art. 164. Page 218.

1. 4. -3.
2. 1.37, nearly.
3. $\frac{x}{3}+\frac{y}{4}\pm\frac{z\sqrt{11}}{12}=1$.
4. $58\frac{1}{3}$
5. 8. $1\frac{1}{3}$.
6. $x+5y-4z-1=0$.
7. $x^2+y^2+z^2-(yz+xz+xy)+x+y+z=\frac{1}{2}$.

Art. 165. Page 219.

1. $(4, -4, 2)$; 118° 7′, 61° 53′, 60°.
2. $3x+4y-12z-12=0$.
3. $7x+5y-z-13=0$.
4. $3x-y-z+5=0$.
5. $3x-y+z-12=0$.

Art. 167. Pages 220–221.

1. $7x-y+z-18=0$.
2. $\frac{1}{2}$; impossible; $\frac{7}{3}$.
3. (*a*) $11x-4y+2z-43=0$. (*b*) $8x+3y+5z-36=0$.
4. $y+4z-1=0$; $x+z-5=0$; $4x-y-19=0$.
5. $6x-5y-3z\pm 6\sqrt{15}=0$.
6. $5x-3y-7z-20=0$.
7. $x+7z=2$.
8. $2x+y+2z=2\sqrt[3]{3}$.
9. $5x+3y+z=15$.
10. $3x+9y+z=6$.

Art. 168. Page 222.

1. (*a*) $\frac{x}{3}=\frac{y}{-1}=\frac{z+2}{2}$. (*b*) $\frac{x+1}{3}=\frac{y-3}{-5}=\frac{z-2}{2}$. (*c*) $x=2$, $y=-3$.

2. (*a*) $(3, -1, 0)$, $(0, 0, -2)$, $(0, 0, -2)$.
(*b*) $(-4, 8, 0)$, $(\frac{4}{5}, 0, \frac{16}{5})$, $(0, \frac{4}{3}, \frac{8}{3})$.
(*c*) $(2, -3, 0)$, parallel, parallel.

3. (*a*) $\frac{3}{\sqrt{14}}$, $\frac{-1}{\sqrt{14}}$, $\frac{2}{\sqrt{14}}$. (*b*) $\frac{3}{\sqrt{38}}$, $\frac{-5}{\sqrt{38}}$, $\frac{2}{\sqrt{38}}$. (*c*) 0, 0, 1.

4. (*a*) $2(x+1)=2(y-2)=(z+3)\sqrt{2}$.
(*b*) $-2(x+1)=2(y-2)=-(z+3)\sqrt{2}$. (*c*) $2y-x=5$, $z+3=0$.

5. $x=y=z$.

Art. 170. Pages 224–225.

1. $x-2y+4=0$, $x-3z-2=0$, $2y-3z-6=0$; $3x-2y-6z=0$;
$25x-32y-27z+46=0$.

2. $x\sqrt{5}-2y=2\sqrt{5}-10$, $z=7$; $(x-2)\sqrt{3}=z-7$, $y=5$;
$2(y-5)\sqrt{2}=z-7$, $x=2$.

3. $5y-7z+4=0$, $5x-8z-9=0$, $7x-8y-19=0$.

4. $\frac{1}{\sqrt{2}}$, 0, $\frac{1}{\sqrt{2}}$.

5. $(\frac{7}{5}, -\frac{2}{5}, 0)$, $(\frac{8}{5}, 0, \frac{2}{5})$, $(0, -\frac{16}{5}, -\frac{14}{5})$.

6. $\dfrac{x-\frac{7}{5}}{\frac{1}{3}}=\dfrac{y+\frac{2}{5}}{\frac{2}{3}}=\dfrac{z}{\frac{2}{3}}$.

8. $\dfrac{\frac{x-a}{m}}{\sqrt{m^2+n^2+1}}=\dfrac{\frac{y-b}{n}}{\sqrt{m^2+n^2+1}}=\dfrac{\frac{z}{1}}{\sqrt{m^2+n^2+1}}$.

9. (*a*) $\dfrac{5}{\sqrt{29}}, \dfrac{4}{\sqrt{29}}, 0$. (*b*) $0, 1, 0$. (*c*) $\dfrac{-2}{\sqrt{13}}, 0, \dfrac{-3}{\sqrt{13}}$.

10. $\dfrac{x-2}{4}=\dfrac{y}{2}=\dfrac{z+2}{-5}$. **11.** $x=2, y=3$.

Art. 171. Pages 226–227.

1. $(0, 1, -2)$; $(-10, -7, 19)$; $(2, -\frac{13}{4}, \frac{1}{4})$. **4.** $(3, \frac{7}{2}, \frac{1}{3})$.

5. $\dfrac{x-1}{2}=\dfrac{y+6}{-1}=\dfrac{z-2}{6}$. **10.** $8x+y-26z+6=0$.

6. $\dfrac{x+2}{1}=\dfrac{y-3}{4}=\dfrac{z-2}{1}$. **11.** $\dfrac{x-2}{11}=\dfrac{y+1}{8}=\dfrac{z+2}{7}$.

9. $2x+5y-z=19$. **12.** $\dfrac{x-x_1}{A}=\dfrac{y-y_1}{B}=\dfrac{z-z_1}{C}$.

13. $a(x-x_1)+b(y-y_1)+c(z-z_1)=0$.

Art. 183. Page 239.

1. $x^2-y^2-z^2=0$.

2. (*a*) $3x+2y=2$.
 (*b*) $2x^2+2y^2+6yz+6xz+5xy-2x-4y-8z+4=0$.
 (*c*) $6x^2+6y^2-4z^2+15xy-18x-18y+12=0$.
 (*d*) $12y^2+15z^2+12xy-8x-28y+12=0$.

Art. 186. Pages 242–243.

1. $x-3z=k,\ x+3z=m$. **2.** $k=m=\pm\sqrt{2}$.

3. $\dfrac{4}{3}x+\dfrac{\sqrt{194}}{13}z=k,\ \dfrac{4}{3}x-\dfrac{\sqrt{194}}{13}z=m$.

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